Learning sub-Gaussian classes : Upper and minimax bounds

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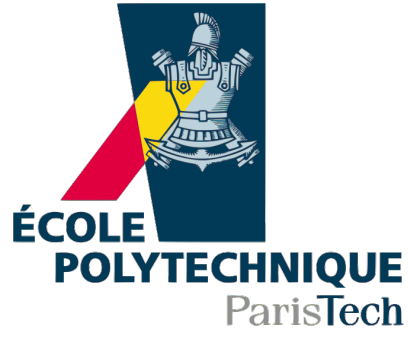

joint work with Shahar Mendelson

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results : Fix $0 < \delta < 1$. With probability greater than $1 - \delta$,

$$R(\hat{f}) \leq \inf_{f \in \mathcal{F}} R(f) + r_N(\mathcal{F}, \delta)$$

where $R(f) = \mathbb{E}(Y - f(X))^2$, $R(\hat{f}) = \mathbb{E}\big[(Y - \hat{f}(X))^2 | (X_i, Y_i)\big]$.

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questions :

a) How large is $r_N(\mathcal{F}, \delta)$? (complexity of $\mathcal{F}$, value of δ,...)

b) Can we do better than ERM ? (minimax results - depending on δ, the complexity structure of $\mathcal{F}$,...)

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3. the Gelfand k-width : $c_k(\mathcal{F}) = \inf_{L:L_2(\mu)\mapsto\mathbb{R}^k} \operatorname{diam}(\mathcal{F} \cap \ker L, L_2(\mu))$.

How are they related ?

$$\sup_{\epsilon>0} \epsilon \log^{1/2} N(\mathcal{F}, \epsilon D) \underset{\text{Sudakov}}{\lesssim} \mathbb{E}\|G\|_{\mathcal{F}} \underset{\text{Dudley}}{\lesssim} \int \log^{1/2} N(\mathcal{F}, \epsilon D) d\epsilon$$

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ex. : $\mathcal{F} = \{\langle \cdot, t\rangle : t \in B_1^d\}$, $X \sim \mu$ is isotropic (i.e. $\mathbb{E}\langle X, t\rangle^2 = \|t\|_{\ell_2^d}^2$) then Sudakov, Carl and [P./T.-J.] are sharp $= \sqrt{\log d}$ but Dudley is not sharp $= (\log d)^{3/2}$.

Assumptions : sub-gaussian framework

1. $\mathcal{F}$ is L-sub-Gaussian : $\forall f, g \in \mathcal{F} \cup \{0\}$,

$$\|f - g\|_{\psi_2(\mu)} \leq L \|f - g\|_{L_2(\mu)}$$

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4. $\mathcal{F} - \mathcal{F}$ is star-shaped around 0 ($[f - g, 0] \subset \mathcal{F} - \mathcal{F}, \forall f, g \in \mathcal{F}$).

Theorem [L.& Mendelson] : sharp oracle inequality for ERM in Sub-Gaussian framework

1. If $\sigma \geq c_3 r_N^*$ then with probability at least $1 - 4\exp(-c_4 N \sigma^{-2}(s_N^*)^2)$,

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Classical fixed points - two main streams

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>1996 Fixed points were associated to the expected supremum of the empirical process (indexed by localized classes) or weighted, symmetrized version, ... : [Massart, Saint Flour 2003] [Koltchinksii, Saint Flour 2008], [Bartlett, Mendelson, PTRF06], [Blanchard, Bousquet, Massart] :

$$\text{residue} = \inf\left\{ s > 0 : \mathbb{E} \sup_{\{f \in \mathcal{F} : P\mathcal{L}_f \leq s\}} |(P - P_N)\mathcal{L}_f| \leq c_0 s \right\}.$$

2 regimes for the noise - 2 statistical complexities - 2 empirical processes

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- the noise $\|Y - f_{\mathcal{F}}^*(X)\|_{\psi_2} = \sigma$ is a source of complexity because *it is a noise !* : the values $f_{\mathcal{F}}^*(X_i)$ are hidden by the "noise" $Y_i - f_{\mathcal{F}}^*(X_i)$.

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Decomposition of the excess loss function :

$$\begin{aligned}\mathcal{L}_f(x,y) = (\ell_f - \ell_{f_{\mathcal{F}}^*})(x,y) &= \big(y - f(x)\big)^2 - \big(y - f_{\mathcal{F}}^*(x)\big)^2 \\ &= (f - f_{\mathcal{F}}^*)^2(x) + 2(y - f_{\mathcal{F}}^*(x))(f_{\mathcal{F}}^* - f)(x)\end{aligned}$$

2 regimes for the noise - 2 statistical complexities - 2 empirical processes

1. The quadratic process $((P - P_N)(f - f_{\mathcal{F}}^*)^2)_{f \in \mathcal{F} \cap rD}$.
   [Mendelson-Pajor-Tomczak] : w.h.p.

$$\sup_{h \in \mathcal{H}} \left| \frac{1}{N} \sum_{i=1}^{N} h^2(X_i) - \mathbb{E}h^2 \right| \lesssim \left(d_{\psi_2}(\mathcal{H}) \frac{\mathbb{E}\|G\|_{\mathcal{H}}}{\sqrt{N}} + \frac{(\mathbb{E}\|G\|_{\mathcal{H}})^2}{N} \right).$$

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$$\sup_{h \in \mathcal{H}} \left| \frac{1}{N} \sum_{i=1}^{N} \xi_i h(X_i) - \mathbb{E}\xi h(X) \right| \lesssim \|\xi\|_{\psi_2} \frac{\mathbb{E}\|G\|_{\mathcal{H}}}{\sqrt{N}}.$$

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$$\sup_{h \in \mathcal{H}} \left| \frac{1}{N} \sum_{i=1}^{N} \xi_i h(X_i) - \mathbb{E}\xi h(X) \right| \lesssim \|\xi\|_{\psi_2} \frac{\mathbb{E}\|G\|_{\mathcal{H}}}{\sqrt{N}}.$$

   This measures the statistical complexity coming from the noise $(= \|\xi\|_{\psi_2} = \|Y - f_{\mathcal{F}}^*(X)\|_{\psi_2} = \sigma)$ via s_N^*.

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ERM is minimax in the Gaussian regression model over sub-Gaussian models (for this confidence bound and noise level $\sigma \gtrsim r^*_N$).

ERM is minimax for high confidence but not for constant confidence

Corollary

In the Gaussian regression model with respect to a sub-Gaussian model for the confidence $1 - 4\exp(-c_4 N\sigma^{-2}(s_N^*)^2)$ *and for a noise level* $\sigma \gtrsim r_N^*$, *ERM is minimax.*

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Theorem (L. & Mendelson)

If $\tilde{f}_N$ is a procedure in the Gaussian model $Y = f^*(X) + W$ ($W \sim \mathcal{N}(0, \sigma^2 I_N)$ ind. of X) such that for every $f^* \in \mathcal{F}$, with probability greater than $3/4$, $R(\tilde{f}_N) \leq \inf_{f \in \mathcal{F}} R(f) + \text{residue}$ then necessarily

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$$\sup_{0 < \epsilon < 2s} \epsilon \log^{1/2} N((\mathcal{F} - \mathcal{F}) \cap 2sD, \epsilon D) \lesssim \mathbb{E} \| G \|_{(\mathcal{F} - \mathcal{F}) \cap 2sD}.$$

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"Sudakov complexity" of the localized set $(\mathcal{F} - \mathcal{F}) \cap 2sD$ at level s :

$$s \log^{1/2} N((\mathcal{F} - \mathcal{F}) \cap 2sD, sD)$$

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For the small noise regime, we obtain the following minimax bound.

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Denote $f^*(X) = \mathbb{E}[Y|X]$. *For every procedure* $\tilde{f}_N$,

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Similar lower bounds have been obtained in Compressed Sensing by [Donoho, IEEE06] or [Cohen, Dahmen, DeVore, JAMS09].

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If "Sudakov is sharp at the level q_N^*" :

$$q_N^* \log^{1/2} N((\mathcal{F}-\mathcal{F})\cap 2q_N^* D, q_N^* D) \sim \mathbb{E}\|G\|_{(\mathcal{F}-\mathcal{F})\cap 2q_N^* D}$$

then upper and lower bounds match and therefore ERM is minimax in the Gaussian model for any subgaussian model for both exponentially large and constant confidences.

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If "Pajor/Tomczak-Jaegermann is sharp at level N" (for some $f_0^* \in \mathcal{F}$) :

$$\sqrt{N} c_N((\mathcal{F} - f_0^*) \cap r_N^* D) \sim \mathbb{E}\|G\|_{r_N^* D \cap (\mathcal{F}-\mathcal{F})}$$

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$$\mathbb{E}\|G\|_{(\mathcal{F}-\mathcal{F})\cap sD} = \mathbb{E} \sup_{\|A\|_{max}\le R; \|A\|_F \le s\sqrt{pq}} \langle \mathfrak{G}, A\rangle$$

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Complexity is important but geometry is even more important

Theorem

Let $X \sim \mu$. Let $\mathcal{F} \subset L_2(\mu)$ be locally compact. The following are equivalent :

i) *for any real valued random variable $Y \in L_2$,*
$\exists f_{\mathcal{F}}^* \in \operatorname{argmin}_{f \in \mathcal{F}} \mathbb{E}(Y - f(X))^2$ and for every $f \in \mathcal{F}$,

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$\Longrightarrow$ the shape of the model really matters in Learning theory.

minimax results for high confidence bounds in $Y = f^*(X) + W$ - sketch of proof

$\tilde{f}_N$ a procedure such that, for every $f^* \in \mathcal{F}$, with probability greater than $1 - \delta$, $R(\tilde{f}_N) \leq \inf_{f \in \mathcal{F}} R(f) + \text{residue}$ i.e. $\|\tilde{f}_N - f^*\|^2_{L_2(\mu)} \leq \text{res}$.

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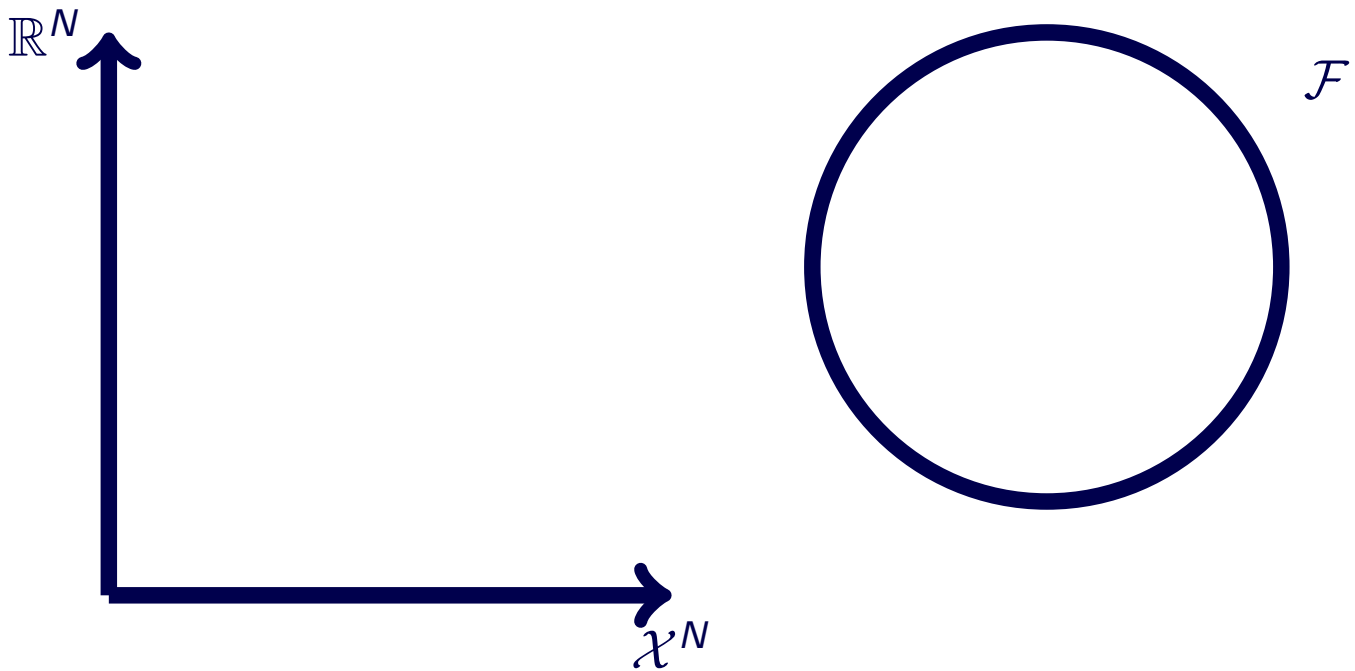

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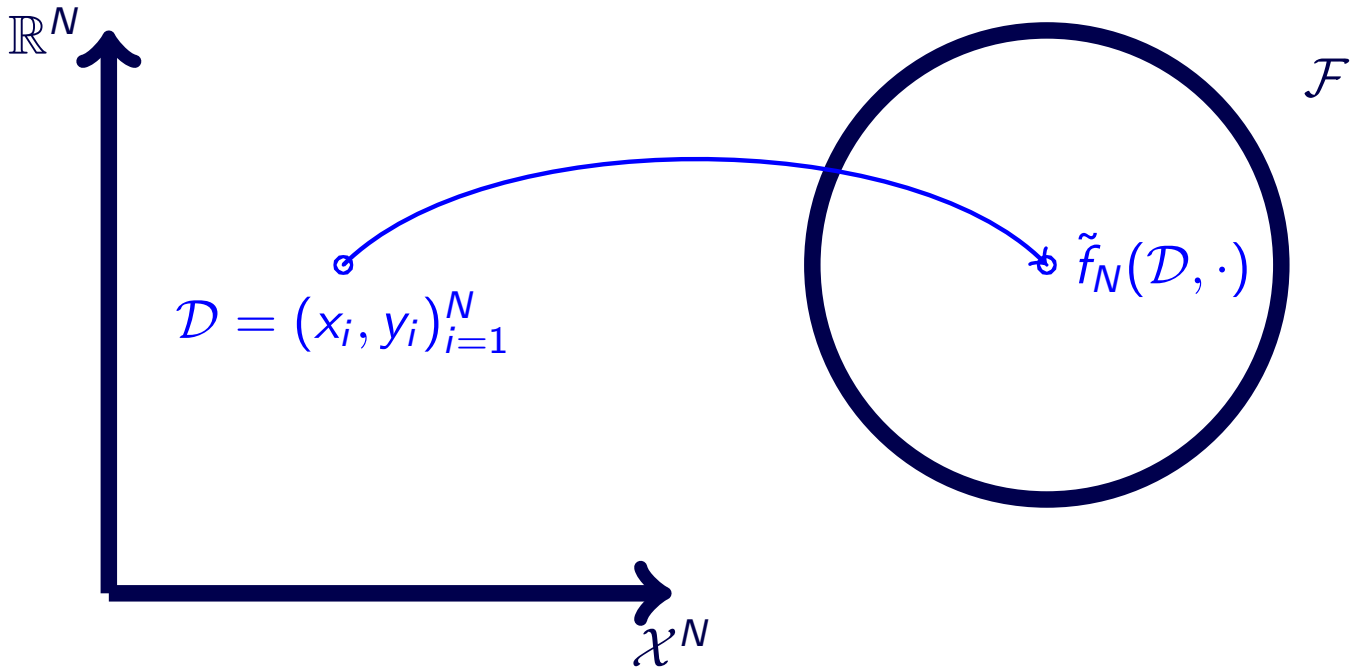

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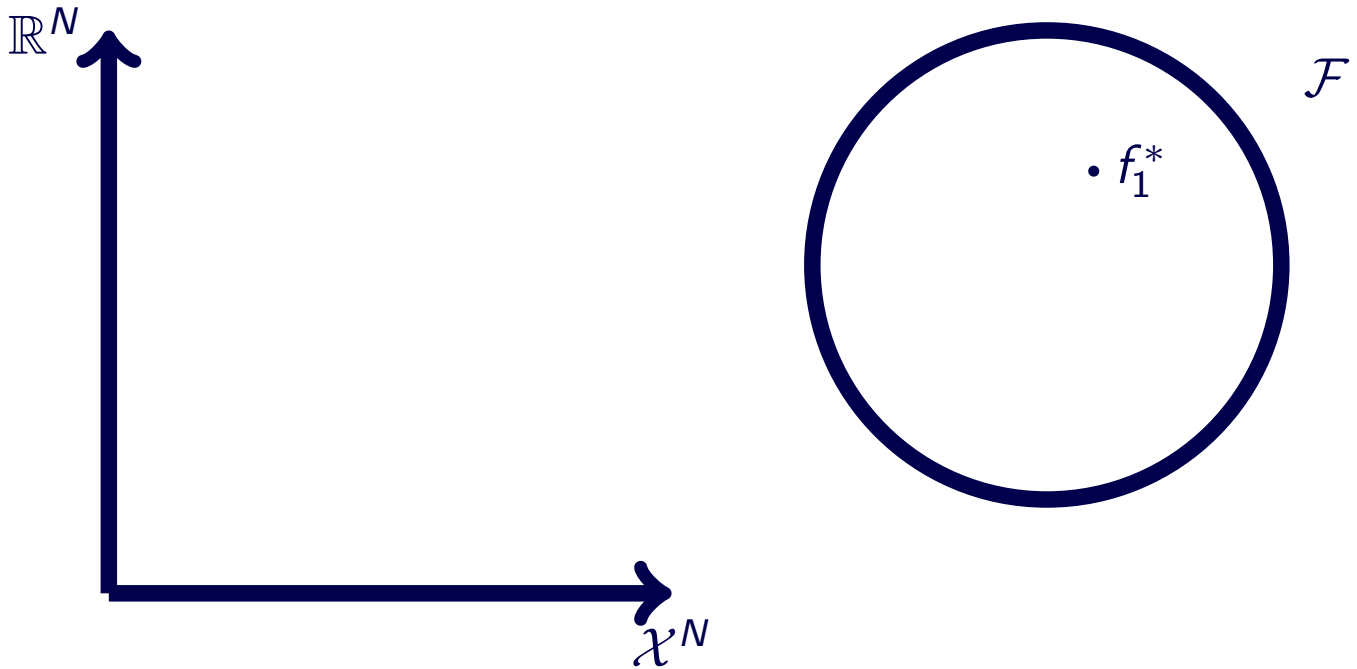

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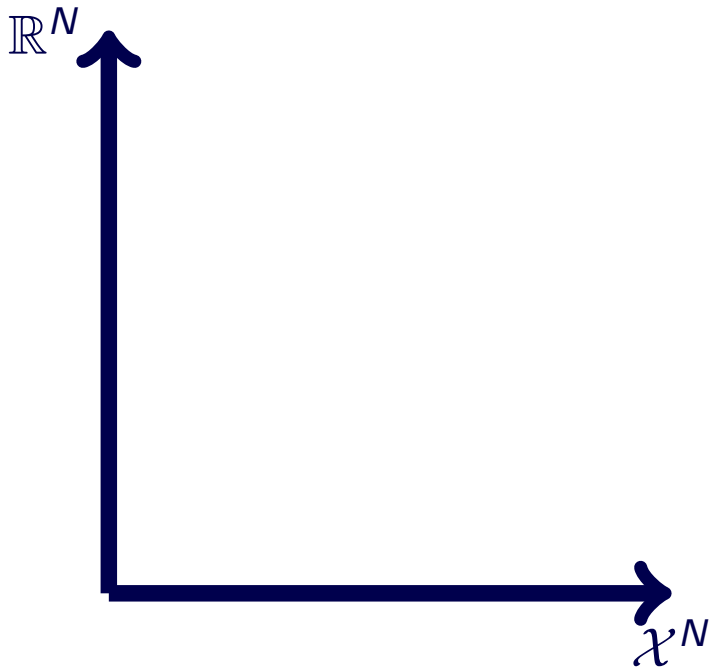

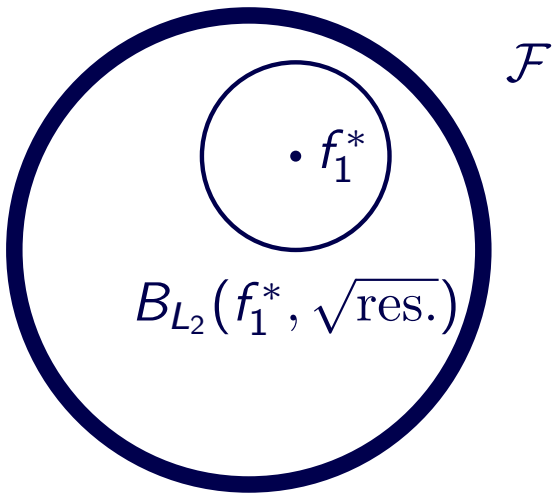

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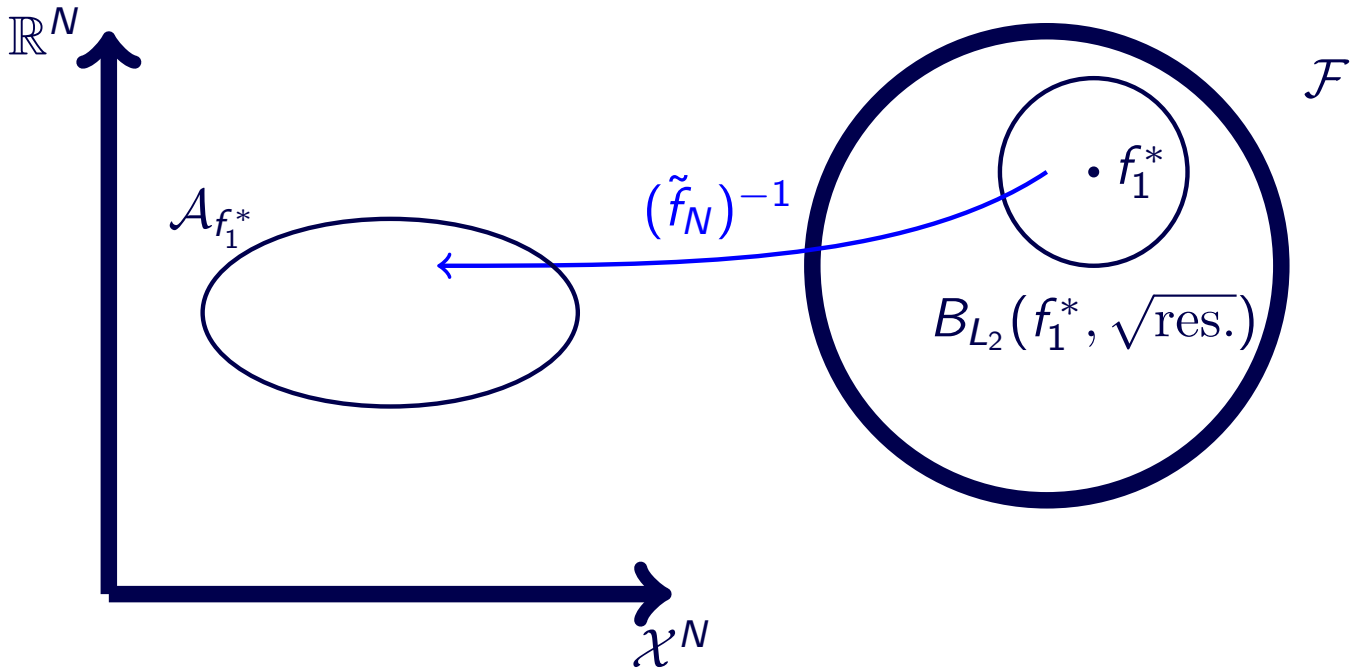

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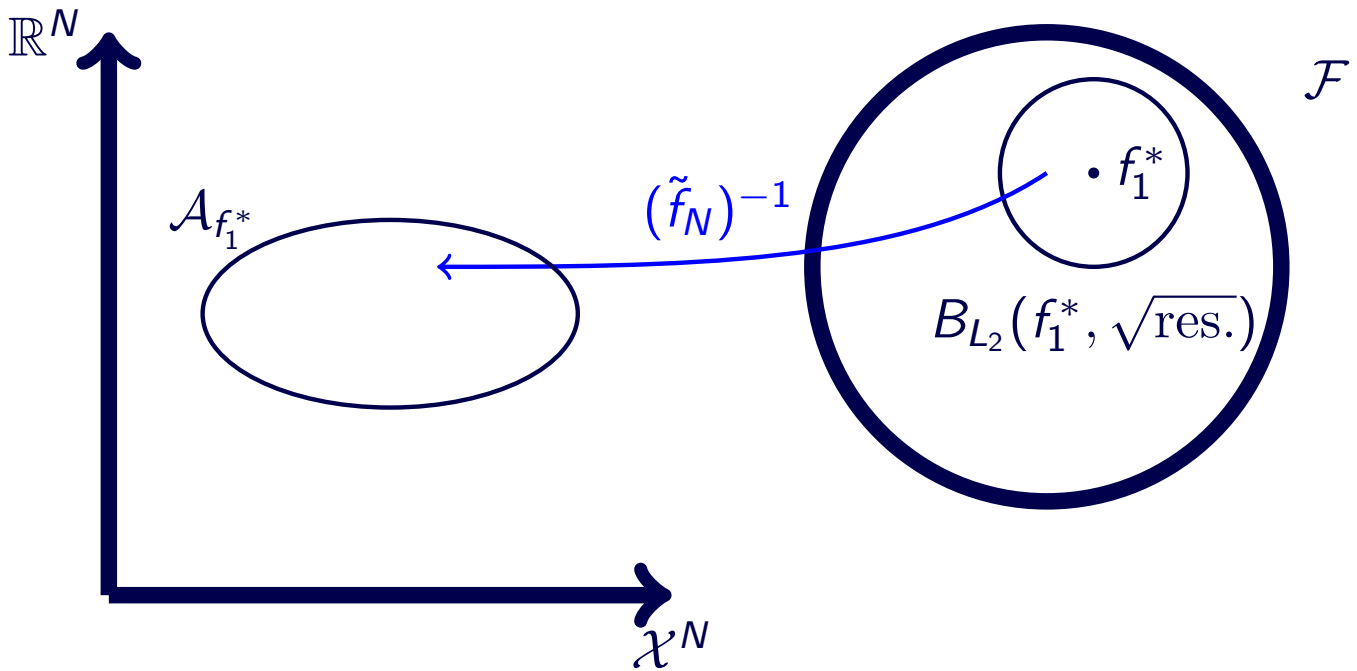

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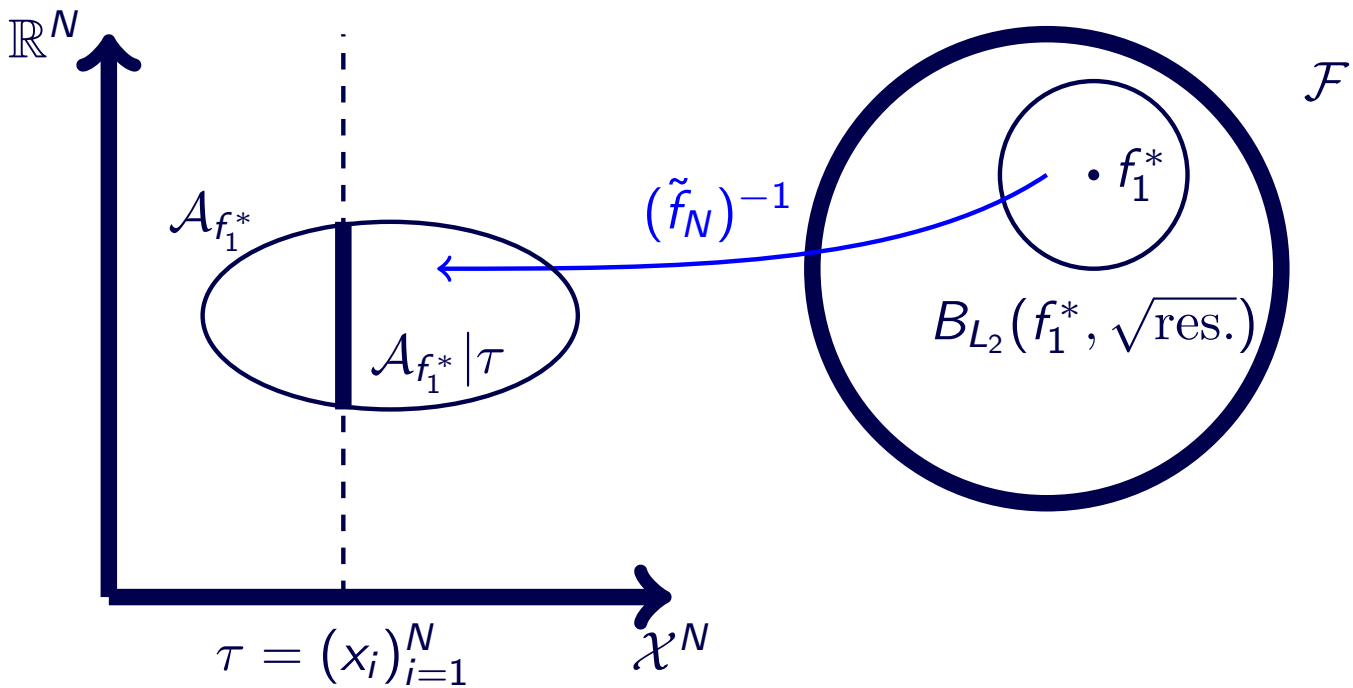

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minimax results for high confidence bounds in $Y = f^*(X) + W$ - sketch of proof

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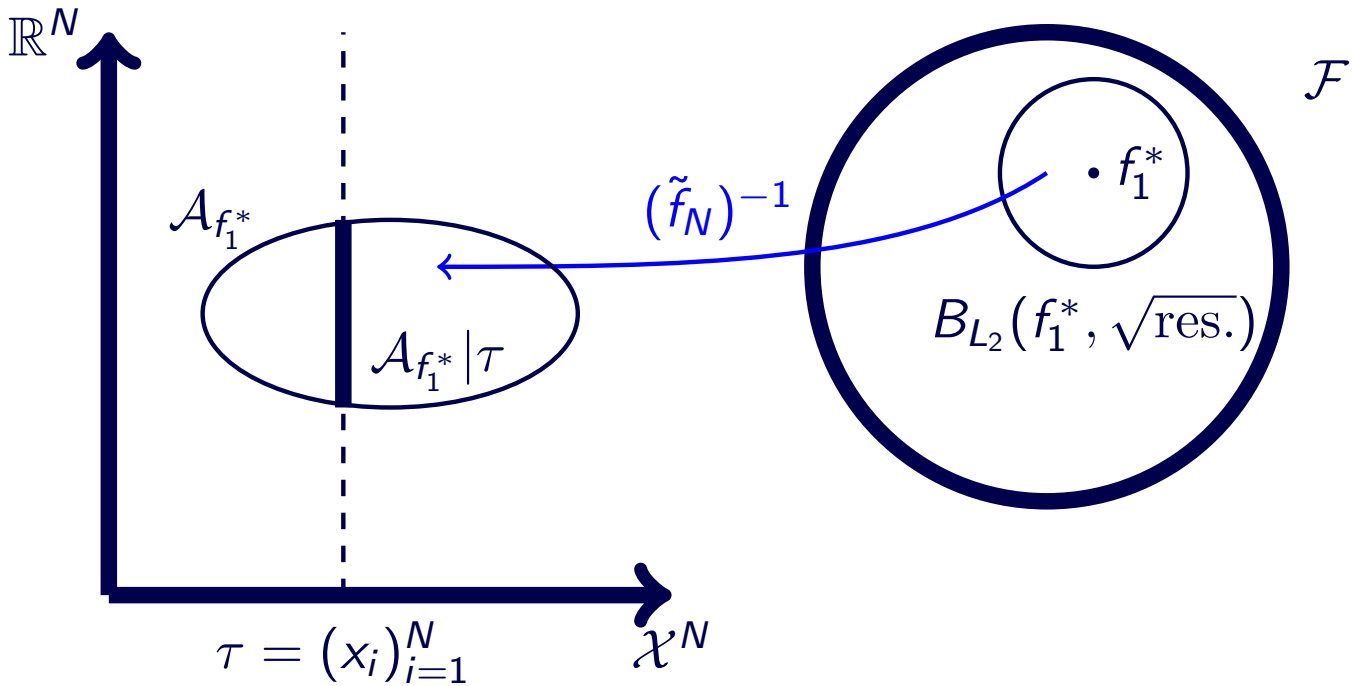

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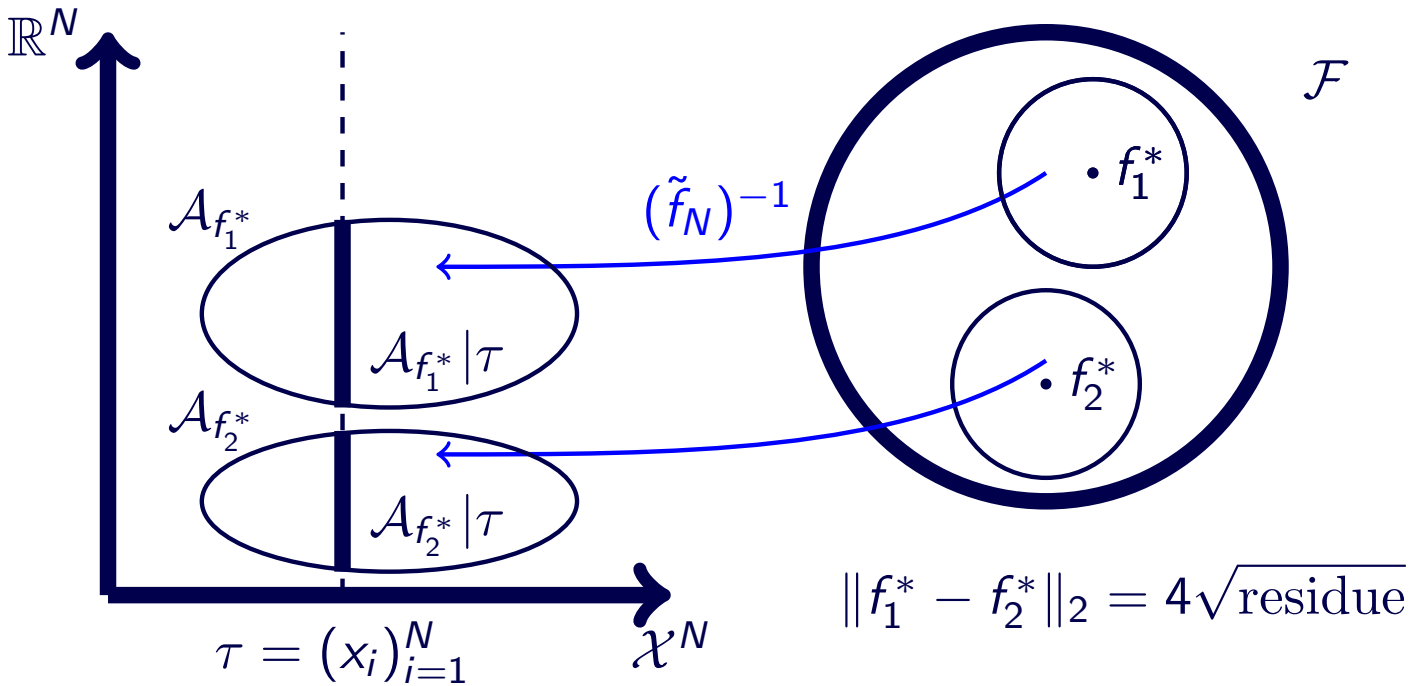

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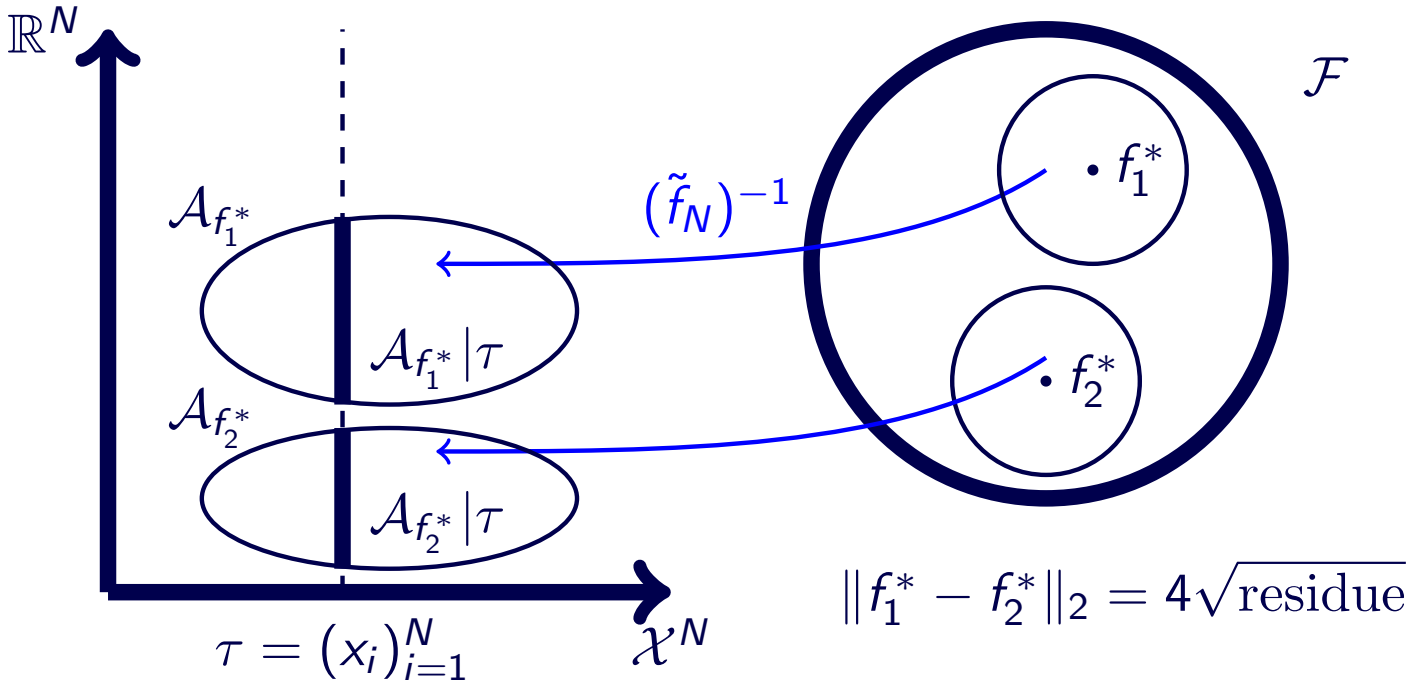

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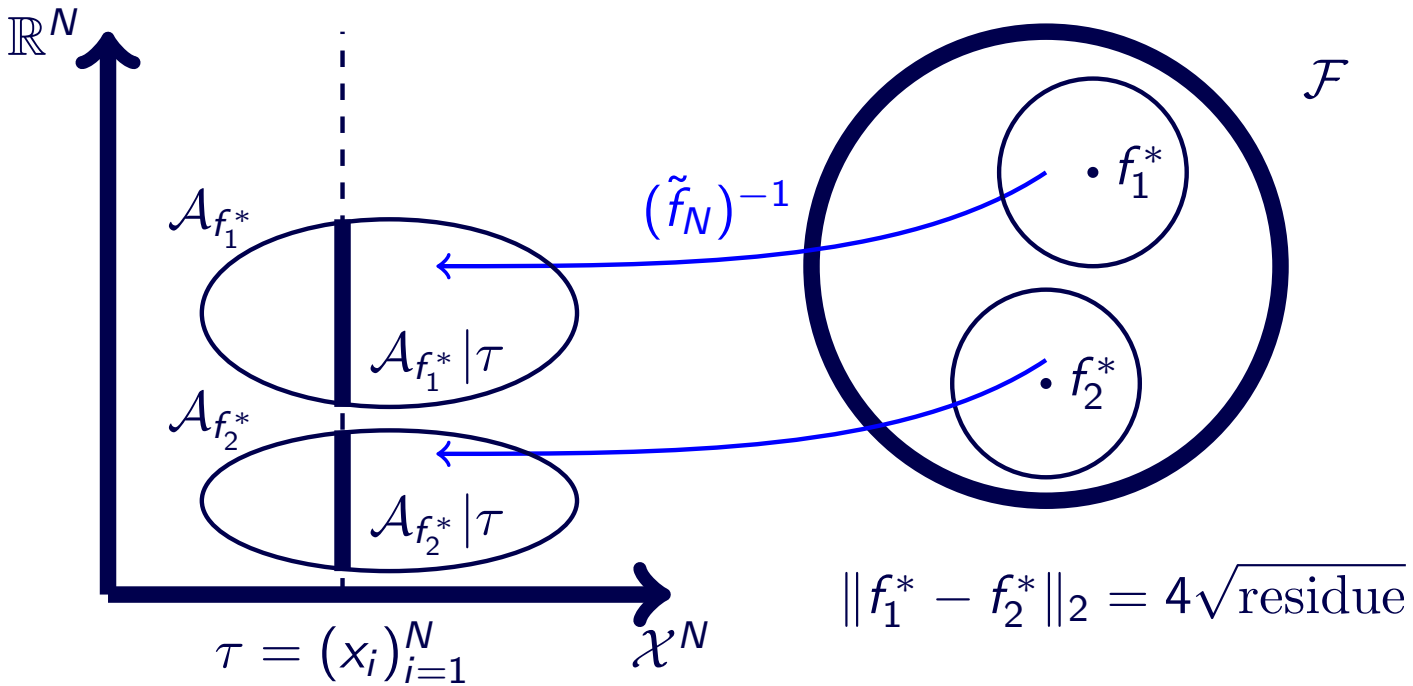

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Theorem ("Gaussian shift theorem". Li & Kuelbs, 98)

Let $\nu \sim \mathcal{N}(0, I_N)$. Let $H_+ = \{x \in \mathbb{R}^N : \langle x, w \rangle \geq b\}$ for some $w \in \mathbb{R}^N, b \in \mathbb{R}$. Let $B \subset \mathbb{R}^N$ such that $\nu(H_+) = \nu(B)$. Then,

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If $\nu_u \sim \mathcal{N}(u, \sigma^2 I_N)$ and $\nu_v \sim \mathcal{N}(v, \sigma^2 I_N)$ then

$$\nu_u(A) \geq 1 - \Phi\big(\Phi^{-1}(1 - \nu_v(A)) + \|u - v\|_{\ell_2^N}/\sigma\big)$$

for $\Phi(t) = \mathbb{P}[\mathcal{N}(0,1) \leq t]$.

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The result follows for $\delta = 4\exp(-c_4 N\sigma^{-2}(s_N^*)^2)$. ■

Thanks for your attention

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The trade-off is obtained for $t = N(s_N^*)^2$.

1. below $t \leq N(s_N^*)^2$ the probability estimate is damaged (the residue is still $(s_N^*)^2$).
2. above $t \geq N(s_N^*)^2$, the residue is damaged.

Other examples of Gaussian mean widths

1. If $p \geq 2$ then $\ell_*(B_p^d \cap sB_2^d) = \ell_*(sB_2^d) = s\sqrt{d}$.
2. If $p < 2$ then set $1 = 1/p + 1/q$ and put $1/r = 1/2 - 1/q$. For any $d^{-1/r} < s \leq 1$,

$$\ell_*(B_p^d \cap sB_2^d) \sim \begin{cases} \sqrt{q}d^{1/q} & \text{if } 2 < q < \log(2d) \text{ and } s^{-1} \leq c_1^{q/r} d^{1/r} \\ \sqrt{\log\left(2ds^2\right)} & \text{if } q \geq \log(2d) \end{cases}$$

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