Suboptimality of Penalized Empirical Risk Minimization in Classification.

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Motivation.

M prior estimators ('weak' estimators) : $f_1, \ldots, f_M$

n observations : D_n

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n observations : D_n

Aim

Construction of a new estimator which is approximatively as good as the best 'weak' estimator :

$\boxed{\text{Aggregation method}}$ or $\boxed{\text{Aggregate}}$

Examples.

Adaptation :

Observations : D_{m+n}

Estimation : D_m $\longrightarrow$non-adaptive estimators $f_1, \dots, f_M$.

learning : $D_{(n)}$ $\longrightarrow$aggregate $\tilde{f}_n$ (adaptive).

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Estimation :

$\epsilon-$net : $f_1, \ldots, f_M$ (functions)

learning : D_n →aggregate $\tilde{f}_n$.

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$(\mathcal{X}, \mathcal{A})$ a measurable space,

$(X, Y) \sim \pi$ valued in $\mathcal{X} \times \{-1, 1\}$,

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$$(f:\mathcal{X} \longmapsto \mathbb{R}) \to \text{risk } A_0(f) = \mathbb{E}[\phi_0(Yf(X))] \text{ where}$$

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$\phi_0(x) = \mathbb{I}_{(x \leq 0)}$	classical loss or $0-1$ loss
$\phi_1(x) = \max(0, 1-x)$	hinge loss or (SVM loss)
$x \longmapsto \log_2(1+\exp(-x))$	'Logit-Boosting' loss
$x \longmapsto \exp(-x)$	exponential Boosting loss
$x \longmapsto (1-x)^2$	quadratic loss
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$\phi-$risk : $A^{\phi}(f) = \mathbb{E}[\phi(Yf(X))], \quad A^{\phi *} \stackrel{\text{def}}{=} \inf_f A(f) = A(f^{\phi *}),$

excess $\phi-$risk : $A^{\phi}(f) - A^{\phi *}$.

empirical $\phi-$risk : $A_n^{\phi}(f) = \frac{1}{n}\sum_{i=1}^{n} \phi(Y_i f(X_i)).$

Selectors.

$\phi : \mathbb{R} \longmapsto \mathbb{R}$ a loss, $\mathcal{F}_0 = \{f_1, \ldots, f_M\} \subset \mathcal{F}$ a dictionary.

- Empirical Risk Minimization (ERM) :(Vapnik, Chervonenkis...)

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- penalized Empirical Risk Minimization (pERM) :

$$\tilde{f}_n^{ERM} \in \mathrm{Arg} \min_{f \in \mathcal{F}_0} [A_n^{\phi}(f) + \mathrm{pen}(f)],$$

where pen is a penalty function. (Barron, Bartlett, Birgé, Boucheron, Koltchinski, Lugosi, Massart,...)

Aggregation methods with exponential weights.

$\phi : \mathbb{R} \longmapsto \mathbb{R}$ a loss, $\mathcal{F}_0 = \{f_1, \ldots, f_M\} \subset \mathcal{F}$ a dictionary.

- Aggregate with Exponential weights (AEW) :

$$\tilde{f}_{n,T}^{AEW} = \sum_{f \in \mathcal{F}_0} w_T^{(n)}(f) f, \text{ where } w_T^{(n)}(f) = \frac{\exp\left(-nTA_n^{\phi}(f)\right)}{\sum_{g \in \mathcal{F}_0} \exp\left(-nTA_n^{\phi}(g)\right)},$$

T^{-1} : temperature parameter.

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- Cumulative Aggregate with Exponential Weights (CAEW) :(Catoni, Yang,...)

$$\tilde{f}_{n,T}^{CAEW} = \frac{1}{n} \sum_{k=1}^{n} \tilde{f}_{k,T}^{AEW}.$$

Aim of Aggregation(1) : Optimal rate of aggregation.

Definition

$\forall \mathcal{F}_0 = \{f_1, \ldots, f_M\} \subseteq \mathcal{F}$, $\exists \tilde{f}_n$ such that $\forall \pi \in \mathcal{P}$, $\forall n \geq 1$

$$\mathbb{E}\left[A(\tilde{f}_n) - A^*\right] \leq \min_{f \in \mathcal{F}_0} \left(A(f) - A^*\right) + C_0 \gamma(n, M).$$

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$\gamma(n, M)$ is an optimal rate of aggregation and $\tilde{f}_n$ is an optimal aggregation procedure.

Aim of Aggregation(2) : Adaptation.

Definition (Oracle Inequality)

$\forall \mathcal{F}_0 = \{f_1, \ldots, f_M\} \subseteq \mathcal{F}$, $\exists \tilde{f}_n$ such that $\forall \pi \in \mathcal{P}$, $\forall n \geq 1$

$$\mathbb{E}\left[A(\tilde{f}_n) - A^*\right] \leq C \min_{f \in \mathcal{F}_0} (A(f) - A^*) + C_0 \gamma(n, M),$$

where $C \geq 1$.

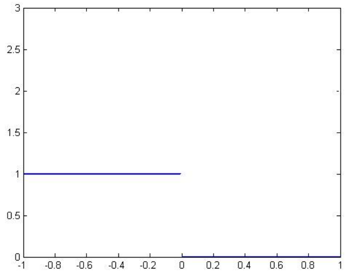
Continuous scale of loss functions.

Classification problem : $A^{\phi}(f) = \mathbb{E}[\phi(Yf(X))], Y \in \{-1, 1\}, X \in \mathcal{X}$.

$$\phi(x) = \phi_h(x) = \begin{cases} (1-h)\phi_0(x) + h\phi_1(x) & \text{if } 0 \leq h \leq 1 \\ (h-1)x^2 - x + 1 & \text{if } h > 1, \end{cases} \qquad \forall x \in \mathbb{R}$$

where $\phi_0(z) = \mathbb{I}_{(z \leq 0)}$ is the $0-1$ loss and $\phi_1(z) = \max(0, 1-z)$ is the hinge loss.

$h = 0$



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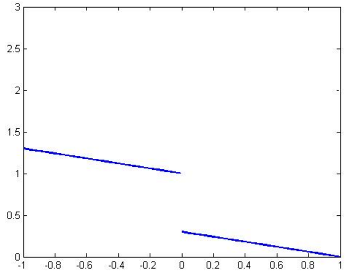
$h = 1/3$



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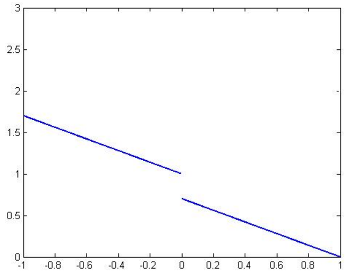

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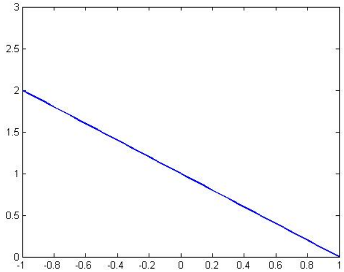

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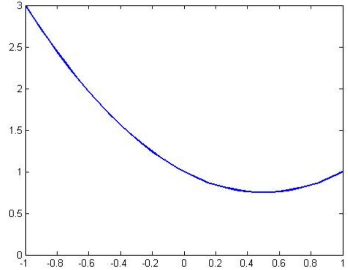

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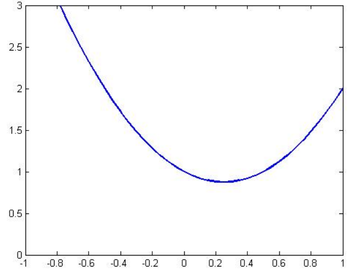
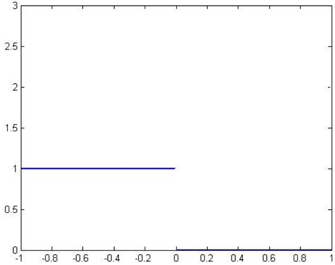


ORA in classification



Loss function	$0 \leq h < 1$	$h = 1$	$h > 1$
Optimal rate of aggregation (ORA)	$\sqrt{\frac{\log M}{n}}$	$\sqrt{\frac{\log M}{n}}$	$\frac{\log M}{n}$
Optimal aggregation procedure	ERM	ERM, AEW, CAEW	CAEW

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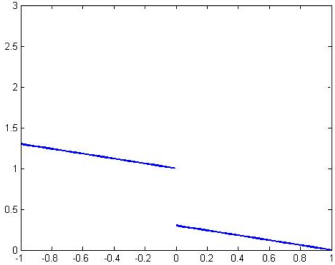

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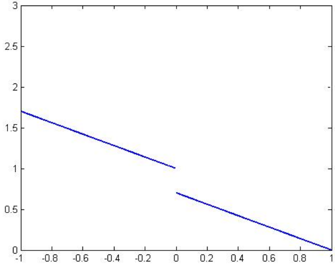

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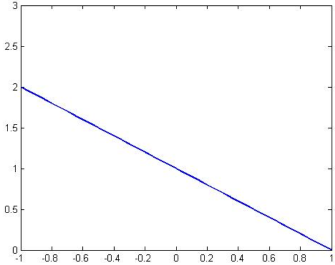

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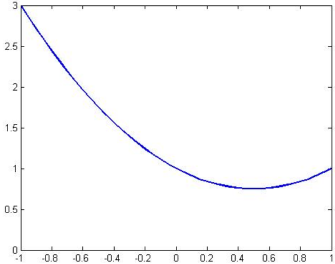

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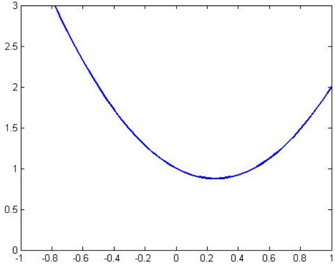

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$$\text{ERM} \quad \longrightarrow \quad \text{CAEW}$$

Question 2 : Do we really need aggregation procedures with exponential weights to achieve the optimal rates of aggregation ?

Question 1. Why there is a breakdown at $h = 1$?

Margin assumption for the loss function ϕ :

The probability measure π satisfies the $\phi-$margin assumption $\phi-\mathrm{MA}(\kappa)$, with margin parameter $\kappa \geq 1$ if

$$\mathbb{E}[(\phi(Yf(X)) - \phi(Yf^{\phi*}(X)))^2] \leq c_\phi (A^\phi(f) - A^{\phi*})^{1/\kappa},$$

for any $f : \mathcal{X} \longmapsto \mathbb{R}$.

cf. Mammen and Tsybakov 99 (discriminant analysis) and Tsybakov 04 (classification).

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$$(\kappa = 1 \Longleftrightarrow \exists h > 0, |2\eta(X) - 1| \geq h)$$

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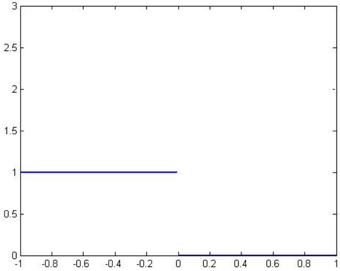

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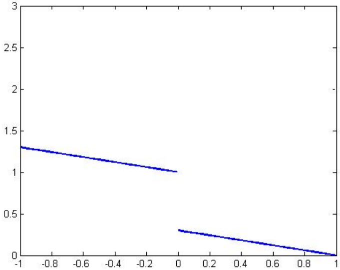

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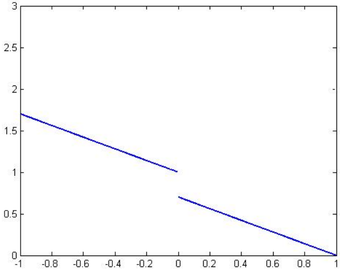
Question 1. Why there is a breakdown at $h = 1$?



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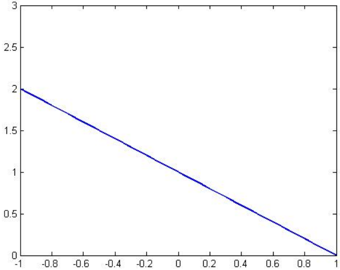

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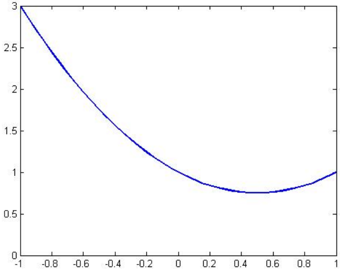

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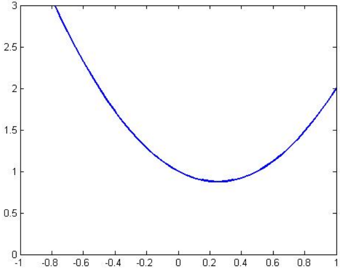

$\kappa = 1$ for any $h > 1$.

Question 2 : Do we really need agg. with exp. weights ?

Theorem (suboptimality of selectors)

For any $M \geq 2$, $\phi : \mathbb{R} \longmapsto \mathbb{R}$ s.t. $\phi(-1) \neq \phi(1)$,
$\exists f_1, \ldots, f_M : \mathcal{X} \longmapsto \{-1, 1\}$ s.t. for any selector $\tilde{f}_n$, $\exists \pi$ s.t.

$$\mathbb{E}\left[A^{\phi}(\tilde{f}_n) - A^{\phi *}\right] \geq \min_{j=1,\ldots,M}\left(A^{\phi}(f_j) - A^{\phi *}\right) + C\sqrt{\frac{\log M}{n}}.$$

Question 2 : Do we really need agg. with exp. weights ?

Theorem (suboptimality of selectors under the margin assumption)

For any $M \geq 2$, $\kappa \geq 1$, $\phi : \mathbb{R} \longmapsto \mathbb{R}$ s.t. $\phi(-1) \neq \phi(1)$, $\exists f_1, \ldots, f_M : \mathcal{X} \longmapsto \{-1, 1\}$ s.t. for any selector $\tilde{f}_n$, $\exists \pi$ satisfying the $\phi_0{-}\mathrm{MA}(\kappa)$ s.t.

$$\mathbb{E}\left[A^{\phi}(\tilde{f}_n) - A^{\phi *}\right] \geq \min_{j=1,\ldots,M}\left(A^{\phi}(f_j) - A^{\phi *}\right) + C\left(\frac{\log M}{n}\right)^{\frac{\kappa}{2\kappa-1}}.$$

$$\sqrt{\frac{\log M}{n}} >> \left(\frac{\log M}{n}\right)^{\frac{\kappa}{2\kappa-1}} >> \frac{\log M}{n}, 1 < \kappa < \infty.$$

Question 2 : Do we really need agg. with exp. weights ?

Suboptimality of Penalized ERM.

For any $M \geq 2$, $\kappa > 1$ and $\phi : \mathbb{R} \longmapsto \mathbb{R}$ s.t. $\phi(-1) \neq \phi(1)$, $\exists f_1, \ldots, f_M : \mathcal{X} \longmapsto \{-1, 1\}$, $\exists \pi$ satisfying the $\phi_0-\mathrm{MA}(\kappa)$ s.t. the pERM aggregate

$$\tilde{f}_n^{pERM} \in \mathrm{Arg} \min_{j=1,\ldots,M} \left(A_n^{\phi}(f_j) + \mathrm{pen}(f_j)\right),$$

where $|\mathrm{pen}(f)| < \frac{1}{6}\sqrt{\frac{\log M}{n}}$, satisfies

$$\mathbb{E}\left[A^{\phi}(\tilde{f}_n^{pERM}) - A^{\phi *}\right] \geq \min_{j=1,\ldots,M} \left(A^{\phi}(f_j) - A^{\phi *}\right) + C\sqrt{\frac{\log M}{n}}$$

if $\sqrt{M \log M} \leq \sqrt{n}/(132 e^3)$, for any integer $n \geq 1$.

Conclusion of optimality

- The margin parameter characterizes the quality of aggregation and estimation in a given model.

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- The margin parameter characterizes the quality of aggregation and estimation in a given model.

- We need convex aggregates to achieve the optimal rate of aggregation for convex losses.