General oracle inequalities for ERM, regularized ERM and penalized ERM with applications to High-Dimensional data analysis

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Joint works with **Stéphane Gaïffas** and **Shahar Mendelson**.

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A quick example : an oracle inequality for the "squared LASSO"

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where $\lambda = \lambda(n, d) = \mathit{polylog}(n, d)$ and $\epsilon > 0$ satisfies, with large probability,

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Question 2 : Why is it possible to achieve a fast $1/n$-residual term without any "RIP -type" assumption ?

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- Aim : construct procedures satisfying some oracle inequalities (no control of the approximation term - we focus on the stochastic term...) – three types of oracle inequalities.

General oracle inequalities for Empirial Risk Minimization

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3. the Empirical Risk Minimization procedure is

$$\hat{f}_n^{(ERM)} \in \operatorname*{argmin}_{f \in F} R_n(f)$$

Three different oracle inequalities. Exemple in aggregation theory.

The ERM over a finite model F w.r.t. the square loss is

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Assume $|Y|, \max_{f\in F}|f(X)| \leq b$ a.s.. For every $x, \epsilon > 0$, with probability greater than $1 - 4\exp(-x)$,

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and its *localized set at level* $\lambda > 0$ is
$$V(H)_\lambda := \{g \in V(H) : \mathbb{E}g \leq \lambda\}$$

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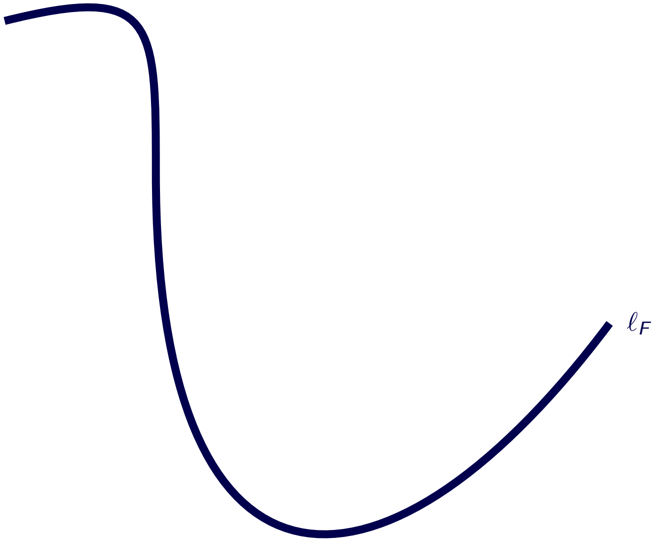

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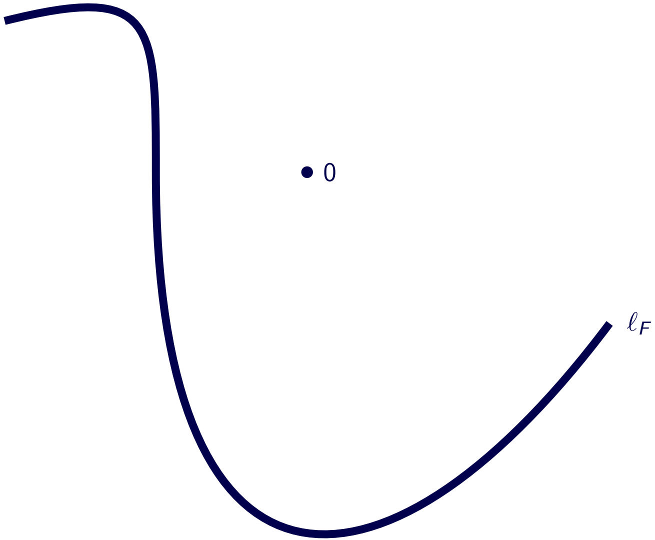

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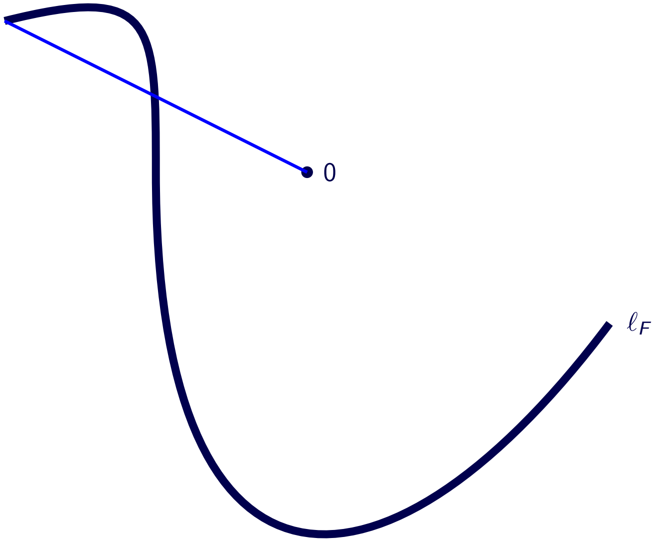

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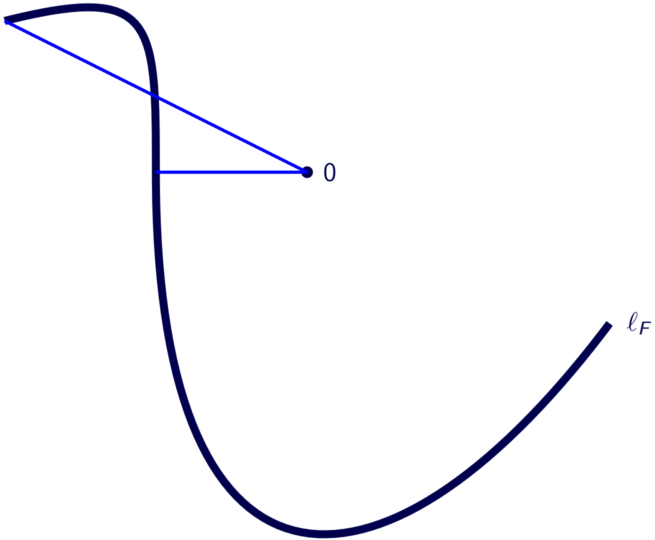

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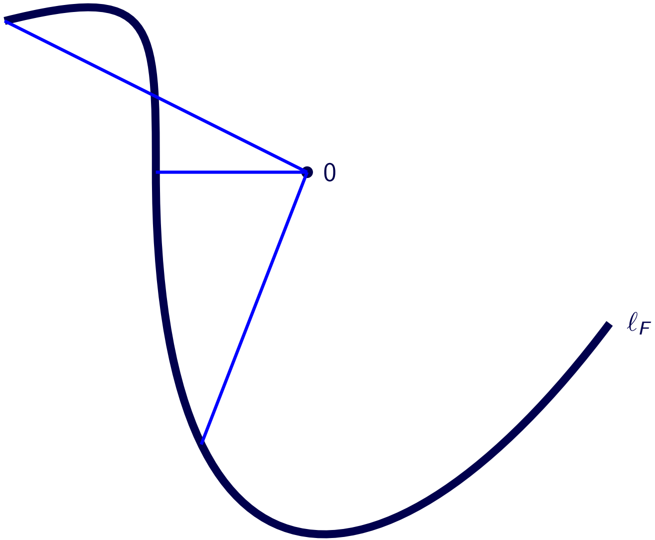

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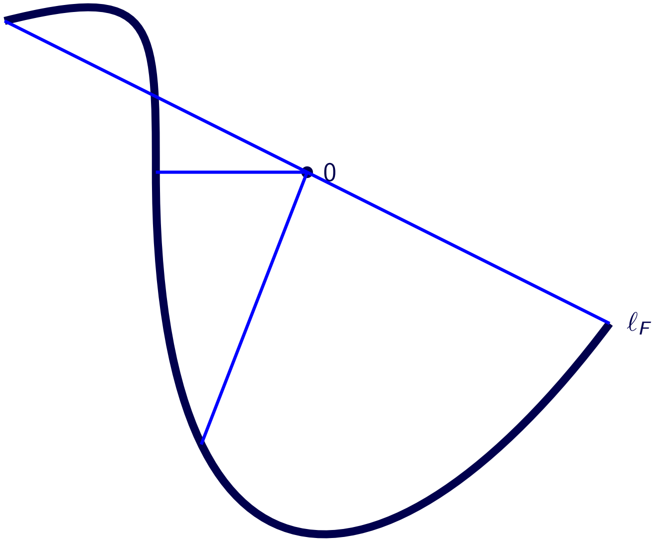

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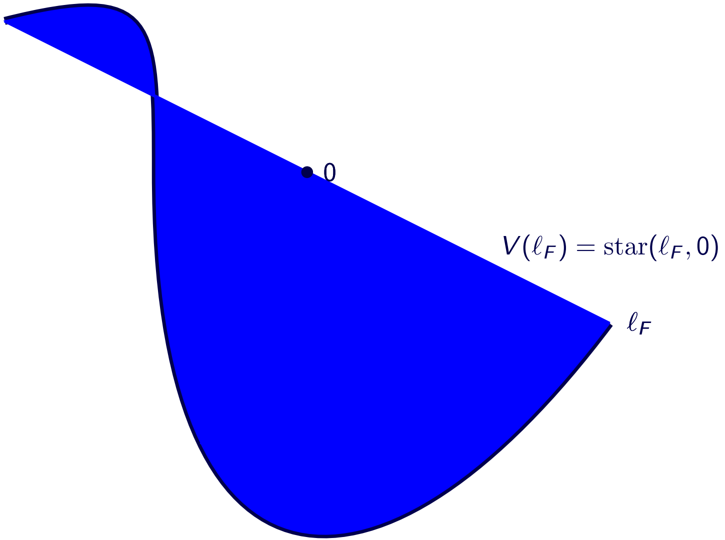

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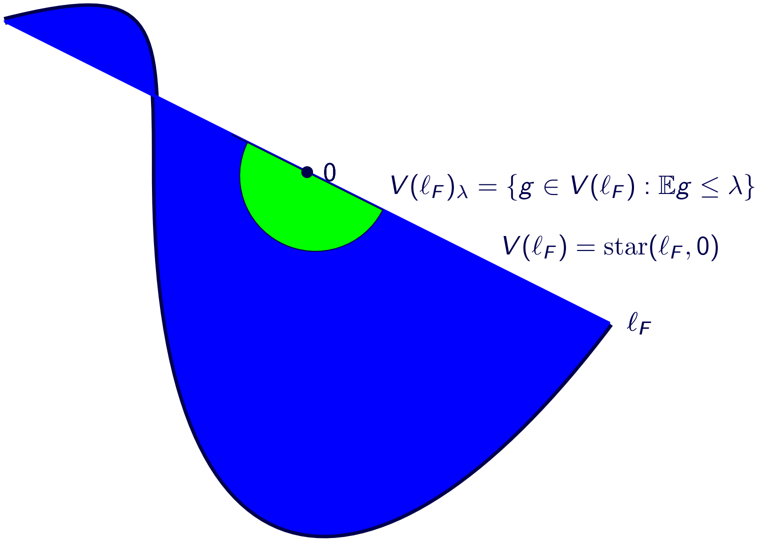

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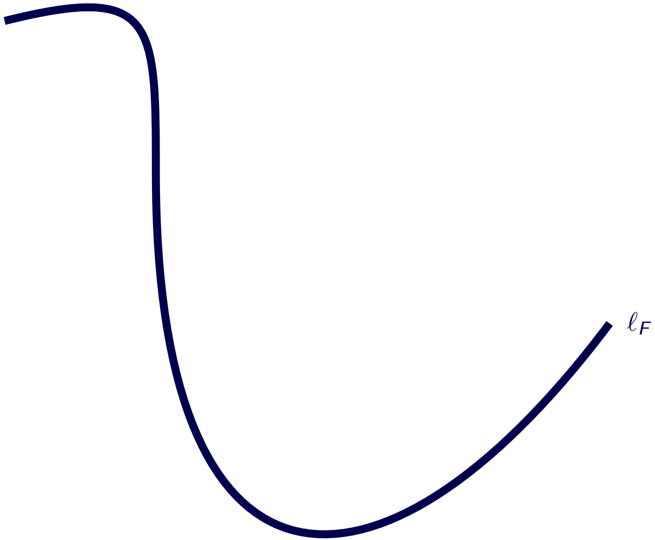

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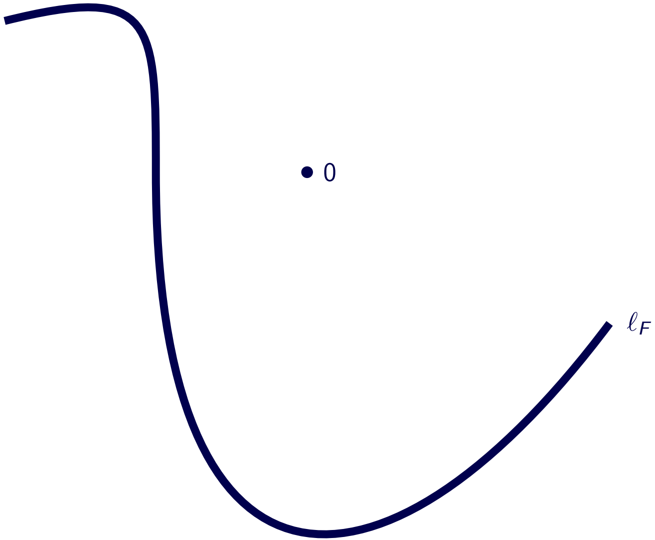

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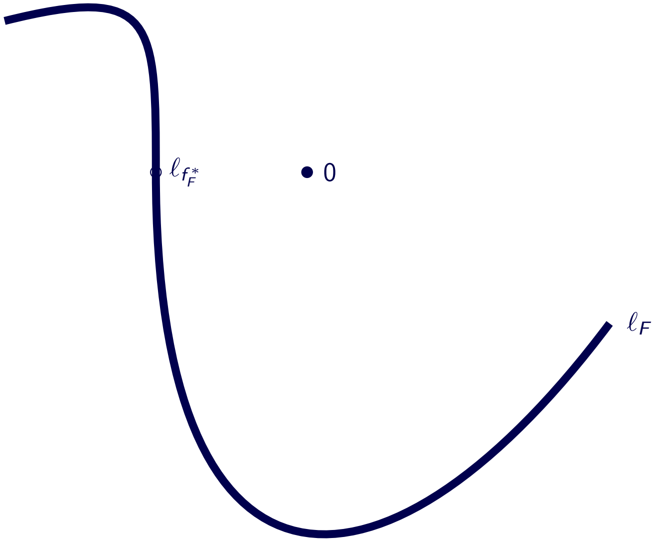

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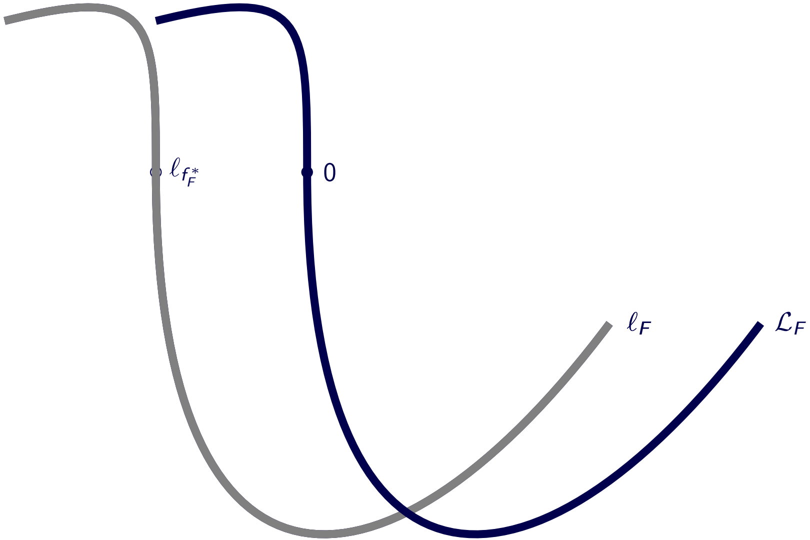

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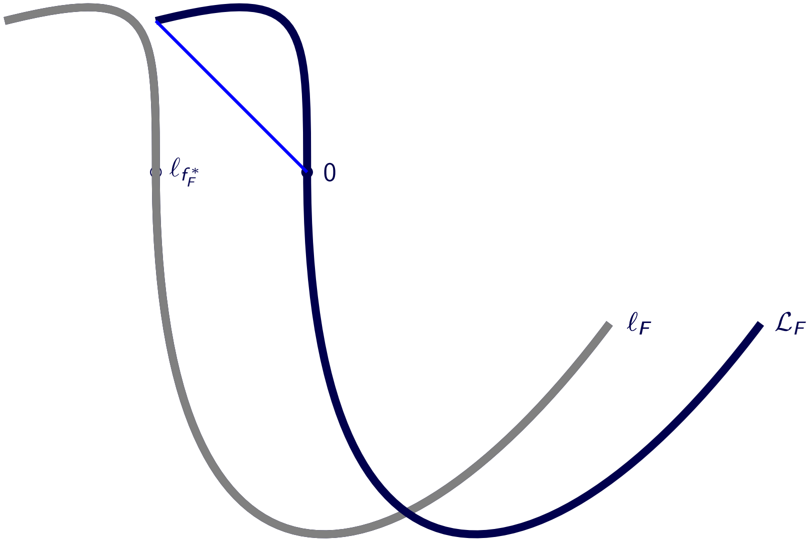

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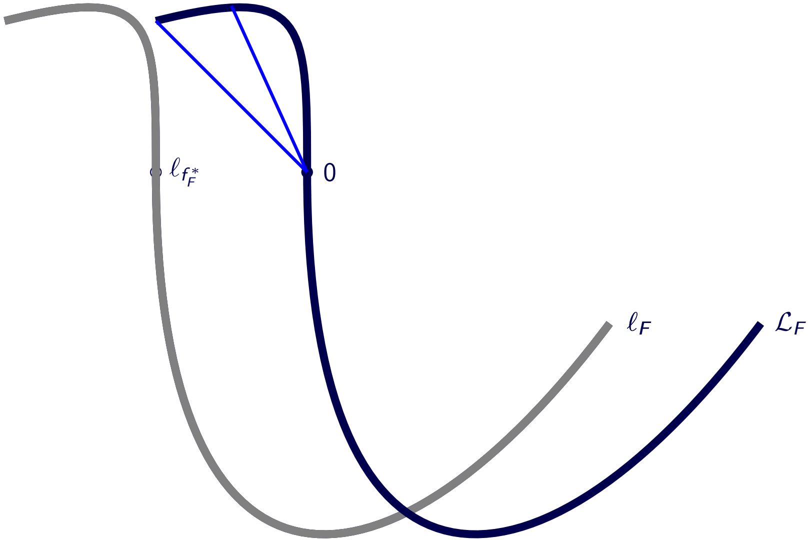

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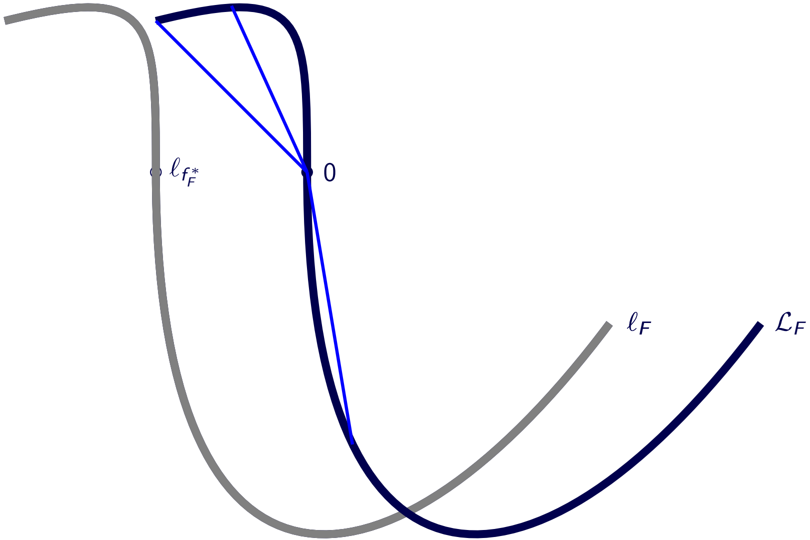

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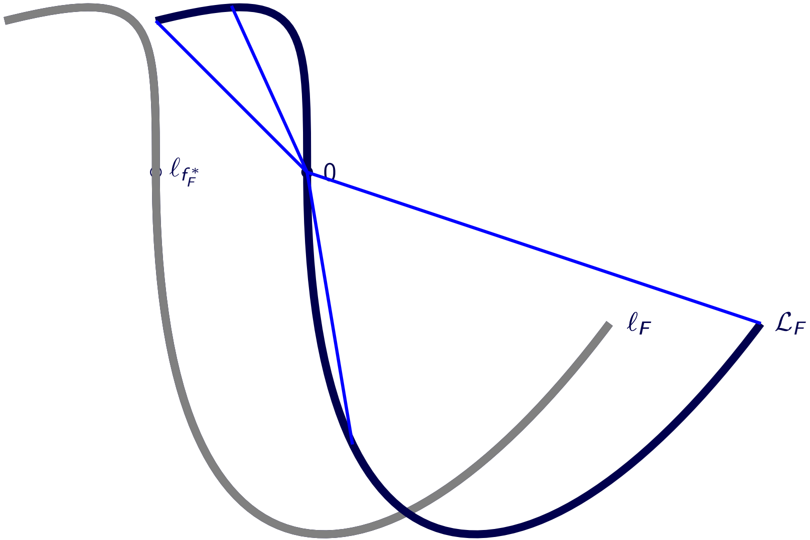

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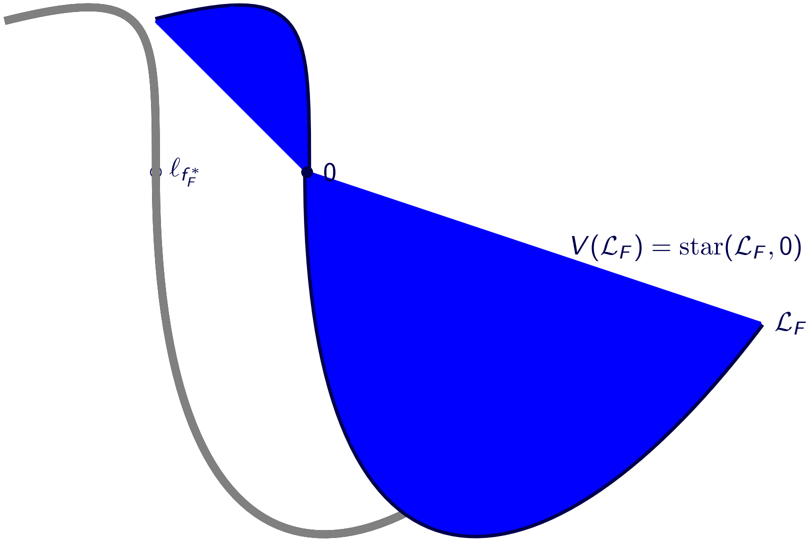

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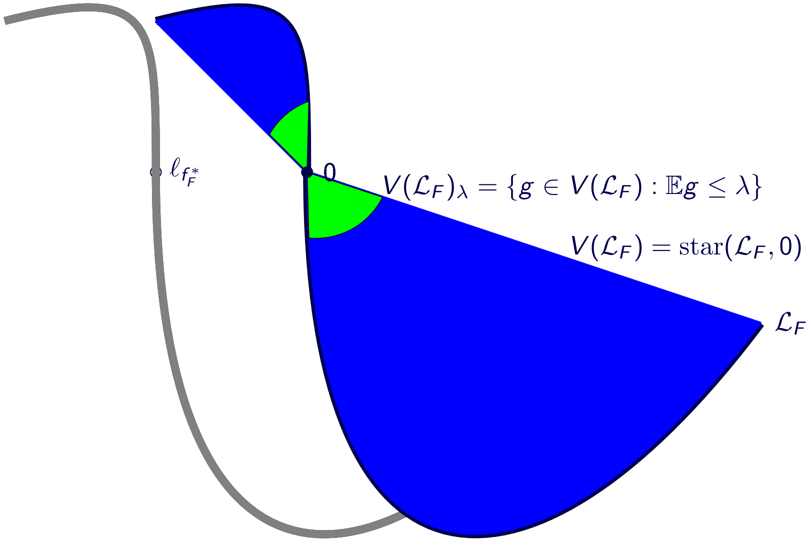

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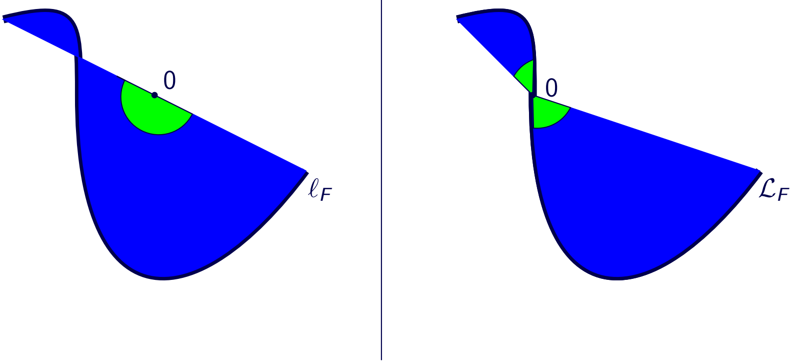

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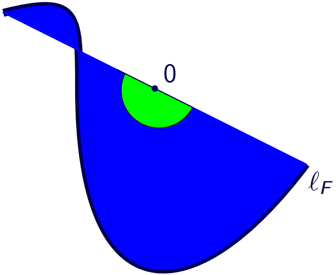

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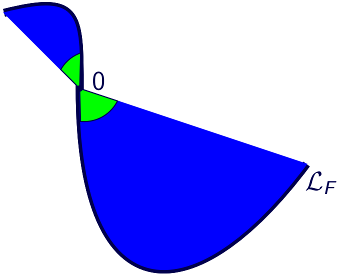

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$$\|P - P_n\|_H := \sup_{h \in H} |Ph - P_n h|$$

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Let F be a class of functions and assume that there exists $B > 0$ such that for every $f \in F$,

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cf. similar results in [Massart and Nédélec], [Koltchinskii],..

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For every function f s.t. $\ell_f \geq 0$ a.s. and $\|\ell_f(Z)\|_{\psi_1} \leq D$ for some $D \geq 1$, we have, for every n,

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Conclusion 1 : In the case of non-exact oracle inequalities, the Bernstein condition for ℓ_F is almost trivially satisfied.

The Bernstein condition of the excess loss class $\mathcal{L}_F$

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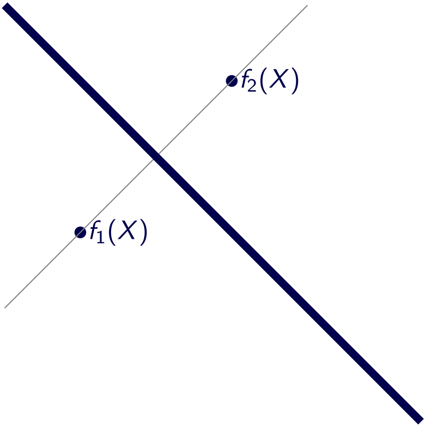

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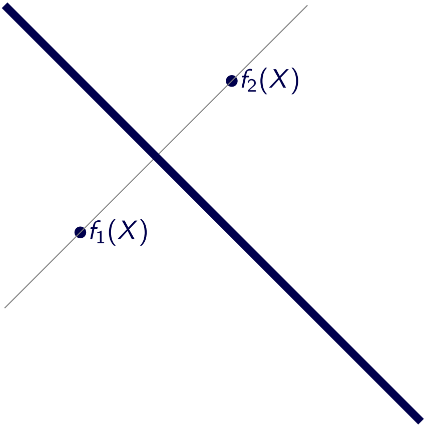

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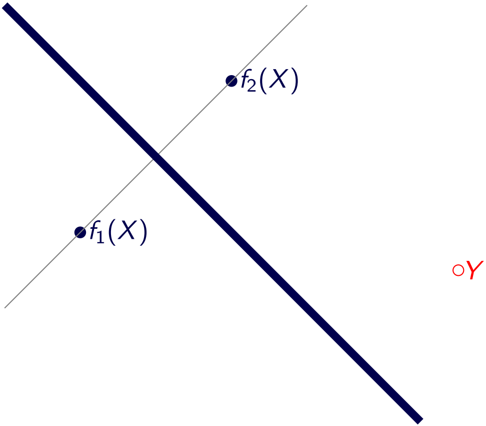

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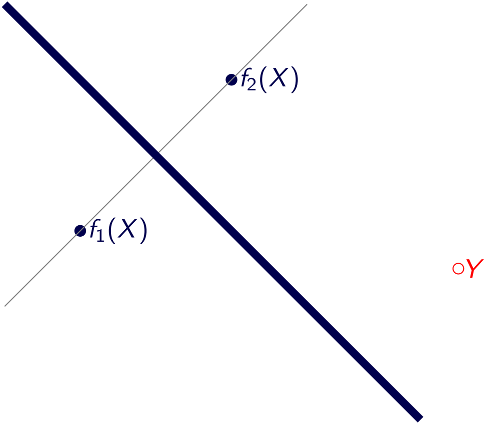

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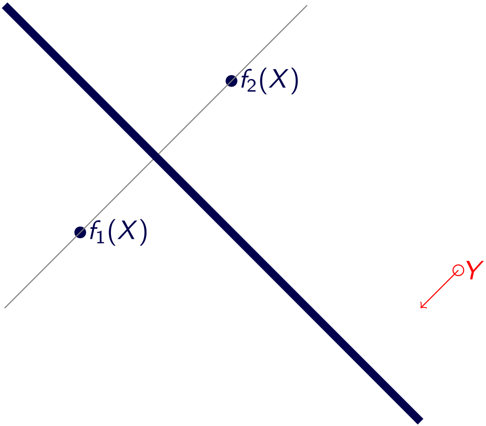

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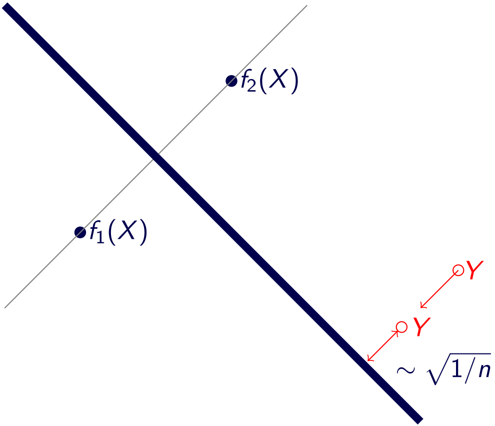

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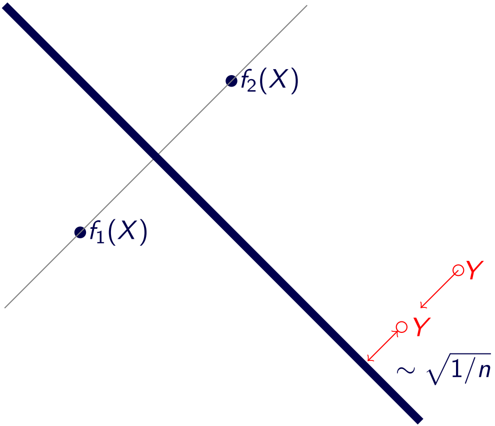

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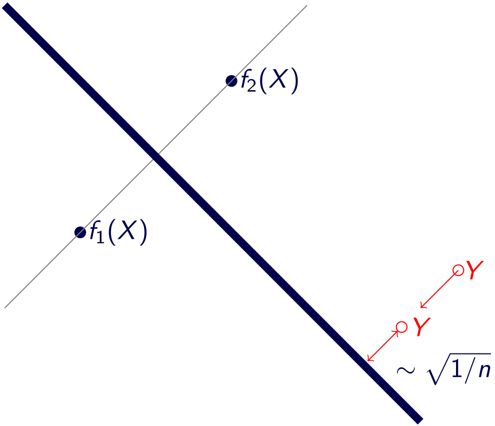

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This explains the gap in the aggregation problem : for this problem, the set of multiple minimizer M_F is never empty. So it is always possible to find a target Y in a "bad" position leading to an excess loss class $\mathcal{L}_F$ with a trivial Bernstein constant ($B \sim \sqrt{n}$) and thus a slow residual term $\sim \sqrt{\mathrm{Comp}(F)/n}$.

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1. When the class F is convex : the Bernstein condition of $\mathcal{L}_F$ is always satisfied (quadratic loss).
2. When the class F is not convex : the ERM is likely to be a suboptimal procedure but there are some possibilities to "improve the geometry" of F : by "starification" (Audibert) or "pre-selection-convexification" (L. and Mendelson).

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The fixed points μ^* and λ^* characterize the isomorphic properties of $\mathcal{L}_F$ and ℓ_F respectively :

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for every $h \in H$ s.t. $Ph \geq \max(\kappa^*, x/n)$ where

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Exact and non-exact oracle inequalities in a general framework - Part 4

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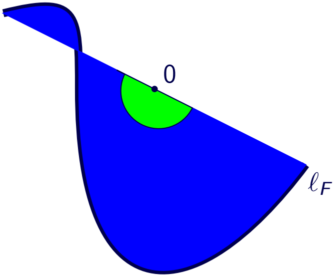

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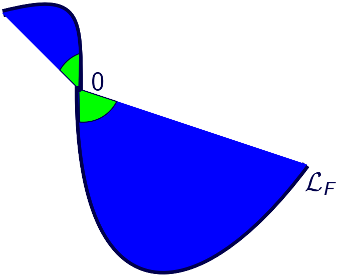

Exact oracle inequalities

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Computation of λ_ϵ^* and μ^* in the case of the Regression model with quadratic loss :

$$\ell_F := \{\ell_f : (y,x) \longmapsto (y - f(x))^2 : f \in F\}$$

and

$$\mathcal{L}_F := \{\mathcal{L}_f : (y,x) \longmapsto (y - f(x))^2 - (y - f_F^*(x))^2 : f \in F\}.$$

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Because $R^* = \inf_{f\in F} R(f) \neq 0$ in general, λ_ϵ^* will be the square of μ^* (of course in some particular cases, we can obtain fast rates for exact oracle inequalities).

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2. The complexities of $V(\mathcal{L}_F)_\lambda$ and $V(\ell_F)_\lambda$ are very different.

Applications to classification

Classification model

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$$\mathcal{L}_f = \ell_f - \ell_{f_F^*} \text{ and } \mathcal{E}_f = \ell_f - \ell_{f^*}.$$

Oracle inequalities in classification

The VC dimension of a class F of $\{0,1\}$-valued functions is

$$V = \max\Big(N : max_{x_1,\cdots,x_N\in\mathcal{X}}\mathrm{Card}\big\{(f(x_1),\dots,f(x_N)) : f\in F\big\} = 2^N\Big).$$

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M.&N. If $P\mathcal{L}_f^2 \leq B\big(P\mathcal{L}_f\big)^\beta, \forall f\in F$ $(0\leq\beta\leq 1)$ Bernstein condition then $\forall x\geq 1$, w.p. $\geq 1-4e^{-x}$,

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Oracle inequalities for regularized ERM

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Regularized Empirical risk minimization - Part 2

Idea : By choosing F, it is implicitly said that we believe that f^* has some properties so that f^* is close to F. But, for a given property on f^* (for instance, smoothness or low-dimensional structure), it is not always possible to construct a class F (with a "reasonable complexity") so that, thanks to this property, f^* will be close to F. In this situation, it is common to introduce a function

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Exact and non-exact oracle inequalities for RERM - Part 1

The choice of the regularizing function $\mathrm{reg}(f) = \lambda \mathrm{crit}^{\alpha}(f)$ is dictated by the complexity of the sequence of models $(F_r)_{r\geq 0}$ where

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(where $R(f_{F_r}^*) = \min_{f\in F_r} R(f)$).

Theorem (L. and Mendelson)

Assume that there are non-decreasing functions ϕ_n and B such that

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Let $0 < \epsilon < 1/2$ and assume that for every $(r,x) \in \mathbb{R}_+ \times \mathbb{R}_+^*$,

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We are given $\mathcal{F}$ and $\mathrm{crit} : \mathcal{F} \longmapsto \mathbb{R}$. We consider the models $(F_r)_{r \geq 0}$:

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Ex. : [Bousquet, Blanchard, Massart] : regularization by $\|\cdot\|_{\mathcal{H}}$ or in [Bartlett, Neeman, Mendelson] : regularization by $\log \|\cdot\|_{\mathcal{H}}$ up to $\|\cdot\|_{\mathcal{H}}^2$.

Applications in matrix completion

Example in matrix completion

Model :

- $Z_1 = (Y_1, X_1), \dots, Z_n = (Y_n, X_n)$: n i.i.d. random variables in $\mathbb{R} \times \mathbb{R}^{m \times T}$;

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1. Candés, Tao, Romberg, Plan, Recht, Fazel, Parillo, Gross,... (Exact reconstruction problem : $Y = \langle X, A_0\rangle$ and often $X \sim \mathrm{Unif}(e_i e_j^\top : 1 \leq i \leq m, 1 \leq j \leq T)$) ;

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2. Tsybakov, Rohde, Koltchinskii, Lounici, Negahban, Wainright, Bach,... (statistical point of view).

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Applications to ℓ_1-regularization

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Oracle inequalitiy for the square LASSO

Let $q \geq 2$. Assume that there exists some constant $c_d > 0$ (which may depend only on d) such that $\|Y\|_{\psi_q}, \|\|X\|_{\ell_\infty^d}\|_{\psi_q} \leq c_d$.For $x > 0$ and $0 < \epsilon < 1/2$, let

$$\lambda(n, d, x) = c_0 c_d^q (\log n)^{(4q-2)/q} (\log d)^2 (x + \log n)$$

and consider the regularized ERM estimator

$$\hat{\beta}_n \in \operatorname*{argmin}_{\beta \in \mathbb{R}^d} \Big(R_n^{(q)}(\beta) + \lambda(n, d, x) \frac{\|\beta\|_{\ell_1}^q}{n\epsilon^2} \Big).$$

Then, with probability greater than $1 - 12\exp(-x)$, the L_q-risk of $\hat{\beta}_n$ satisfies

$$R^{(q)}(\hat{\beta}_n) \leq \inf_{\beta \in \mathbb{R}^d} \Big((1 + 2\epsilon) R^{(q)}(\beta) + c_1 \lambda(n, d, x) \frac{(1 + \|\beta\|_{\ell_1}^q)}{n\epsilon^2} \Big).$$

Oracle inequalities for penalized estimators

Model selection framework

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construction of pen depends on the type of oracle inequality that we want to prove : for any $m \in \mathcal{M}$

$$\ell_m = \{\ell_f : f \in m\}, \quad \mathcal{L}_m = \{\ell_f - \ell_{f_m^*} : f \in m\} \text{ and } \mathcal{E}_m = \{\ell_f - \ell_{f^*} : f \in m\}$$

where we assume that there exists $f_m^* \in \operatorname{argmin}_{f\in m} R(f)$ for any $m \in \mathcal{M}$ (and $f^* \in \operatorname{argmin}_f R(f)$).

Three fixed points

1. For non-exact oracle inequalities : $\forall m \in \mathcal{M}$, for some $0 < \eta < 1/2$,

$$\mathbb{E}\|P_n - P\|_{V(\ell_m)_{\lambda_\eta^*(m)}} \leq (\eta/4)\lambda_\eta^*(m).$$

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2. For non-exact oracle inequalities for the estimation problem : $\forall m \in \mathcal{M}$

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$$\mathbb{E}\|P_n - P\|_{V(\mathcal{L}_m)_{\mu^*_{1/2}(m)}} \leq (1/8)\mu^*_{1/2}(m)$$

where $\mathcal{L}_m = \{\ell_f - \ell_{f^*_m} : f \in m\}$ and $f^*_m \in \mathrm{argmin}_{f \in m} R(f)$.

Non-exact oracle inequalities for the penalized estimator

Assume that there are some functions ϕ_n and B_n such that for every $m \in \mathcal{M}$ and every $f \in m$,

$$\|\max_{1\leq i\leq n}\sup_{f\in m}\ell_f(Z_i))\|_{\psi_1} \leq \phi_n(m) \text{ and } P\ell_f^2 \leq B_n(m)P\ell_f + B_n^2(m)/n.$$

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Let $0 < \eta < 1/2$ and assume that for every $(m, x) \in \mathcal{M} \times \mathbb{R}_+^*$,

$$\rho_n^\ell(m, x) \geq \max\Big(\lambda_\eta^*(m), c_0 \frac{(\phi_n(m) + B_n(m)/\eta)(x+1)}{n\eta}\Big).$$

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$$R(\hat{f}_{\hat{m}}) \leq \frac{1+\eta}{1-\eta}\inf_{m\in\mathcal{M}}\big(\inf_{f\in m}P\ell_f + \mathrm{pen}^\ell(m)\big).$$

Non-exact oracle inequalities for the penalized estimator

$$\mathrm{pen}^{\ell}(m) = \max\left(\lambda_\eta^*(m), c_0 \frac{(\phi_n(m) + B_n(m)/\eta)(x + x_m + 1)}{n\eta}\right) \sim \lambda_\eta^*(m)$$

where

$$\mathbb{E}\|P_n - P\|_{\mathcal{V}(\ell_m)_{\lambda_\eta^*(m)}} \leq (\eta/4)\lambda_\eta^*(m).$$

Oracle inequality for the estimation problem

Assume that there exists $0 < \beta \leq 1$ and some functions ϕ_n and B_n such that for every $m \in \mathcal{M}$ and every $f \in m$,

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$$R(\hat{f}_{\hat{m}}) - R(f^*) \leq \frac{1+\eta}{1-\eta} \inf_{m \in \mathcal{M}} \big(\inf_{f \in m} \big(R(f) - R(f^*)\big) + \mathrm{pen}^{\mathcal{E}}(m)\big).$$

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In the context of the estimation problem, a possible way of penalizing the empirical risk is by the function

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where

$$\mathbb{E}\|P_n - P\|_{V(\mathcal{E}_m)_{\nu_\eta^*(m)}} \leq (\eta/4)\nu_\eta^*(m).$$

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Take $\mathcal{M} = \left(m_n\right)_{n \in \mathbb{N}}$ s.t. $m_0 \subset m_1 \subset m_2 \subset \cdots$.

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Take $\mathcal{M} = \big(m_n\big)_{n\in\mathbb{N}}$ s.t. $m_0 \subset m_1 \subset m_2 \subset \cdots$. Assume that there exists $0 < \beta \leq 1$ and two non-decreasing functions ϕ_n and B_n such that for every $m \in \mathcal{M}$ and every $f \in m$,

$$\|\max_{1\leq i\leq n}\sup_{f\in m}\mathcal{L}_f(Z_i)\|_{\psi_1} \leq \phi_n(m) \text{ and } P\mathcal{L}^2_{m,f} \leq B_n(m)\big(P\mathcal{L}_{m,f}\big)^\beta + B_n^2(m)/n$$

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Let $(x_m)_{m\in\mathcal{M}}$ be a family of positive numbers such that $\sum_{m\in\mathcal{M}}\exp(-x_m) \leq c_1$. Let $x > 0$ and consider the penalty function $\mathrm{pen}^{\mathcal{L}}(m) = (7/2)\rho_n^{\mathcal{L}}(m, x + x_m)$ and the penalized estimator $\hat{f}_{\hat{m}}$ associated with this penalty function.

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$$R(\hat{f}_{\hat{m}}) \leq \inf_{m\in\mathcal{M}} \big(\inf_{f\in m} R(f) + (18/7)\mathrm{pen}^{\mathcal{L}}(m)\big).$$

Exact oracle inequality for the penalized estimator

Therefore, for the exact prediction problem, a possible way of penalizing the empirical risk is by the function

$$\mathrm{pen}^{\mathcal{L}}(m) = c_2 \max\left(\mu^*_{1/2}(m), (B_n(m) + \phi_n(m))\left(\frac{x + x_m}{n}\right)^{\frac{1}{2-\beta}}\right)$$

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and $\mathcal{L}_m = \{\ell_f - \ell_{f^*_m} : f \in m\}$ and $f^*_m \in \mathrm{argmin}_{f \in m} R(f)$.

Thanks ! !