Empirical risk minimization in linear regression and phase recovery

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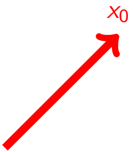
12th November 2013 - Göttingen

joint works with Shahar Mendelson

Two frameworks : linear regression and phase recovery



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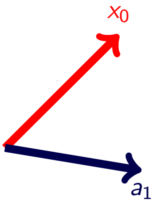

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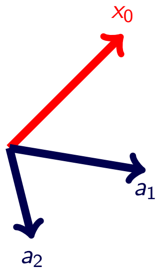

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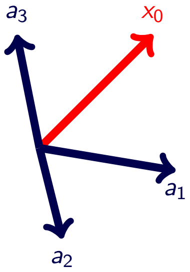

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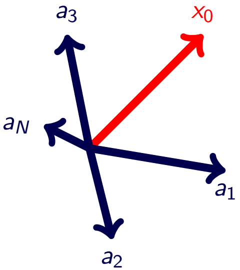

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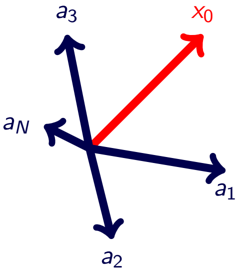

$T \subset \mathbb{R}^d$ and $x_0 \in T$

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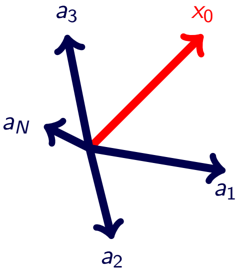

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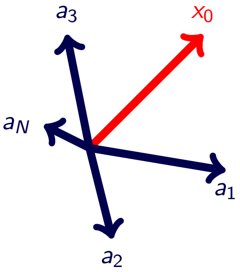

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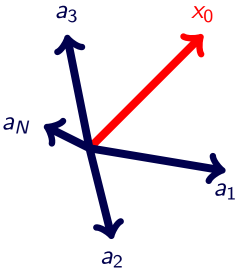

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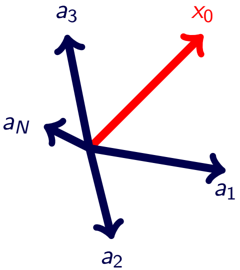

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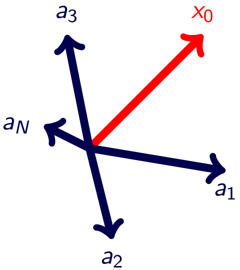

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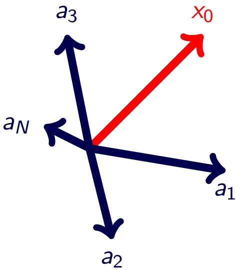

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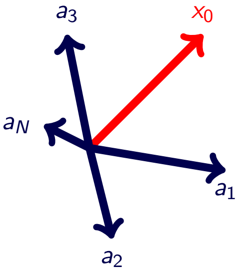

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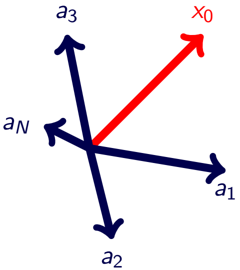

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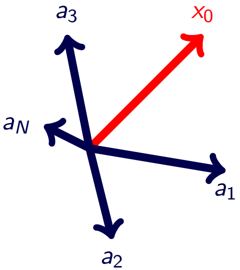

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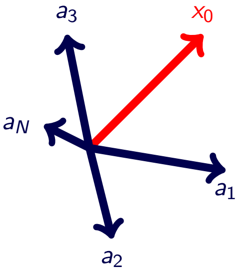

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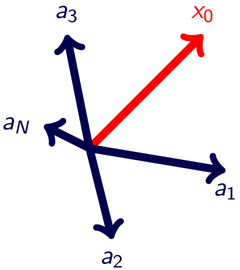

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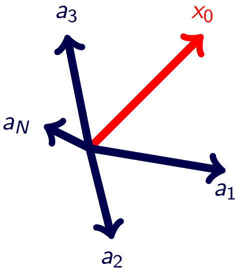

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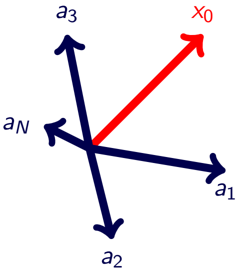

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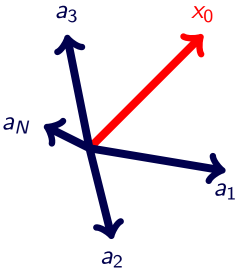

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ex. : Gaussian measurements, Rademacher measurements.

Linear regression

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Ordinary least square estimator - Maximum likelihood estimator

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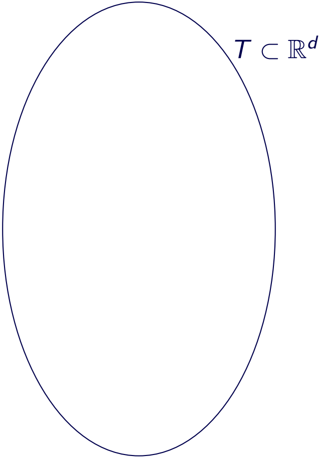

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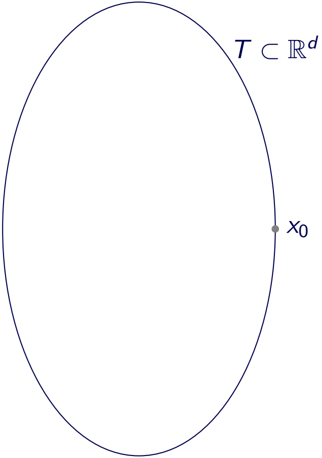

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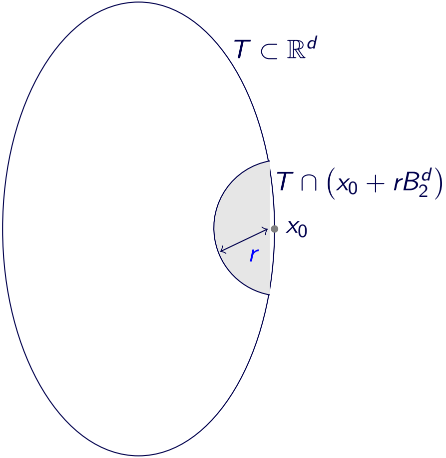

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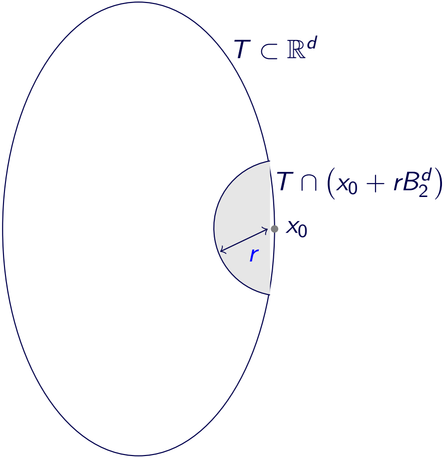

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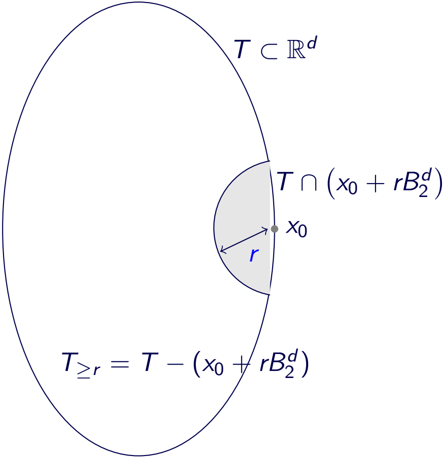

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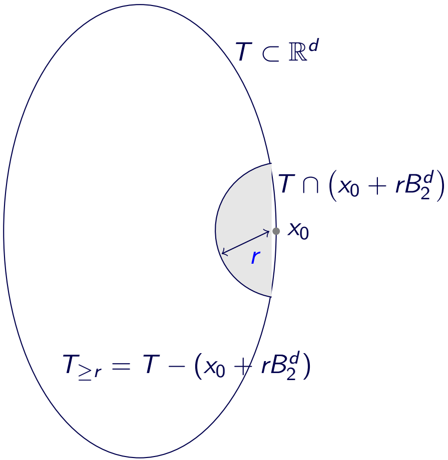

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where

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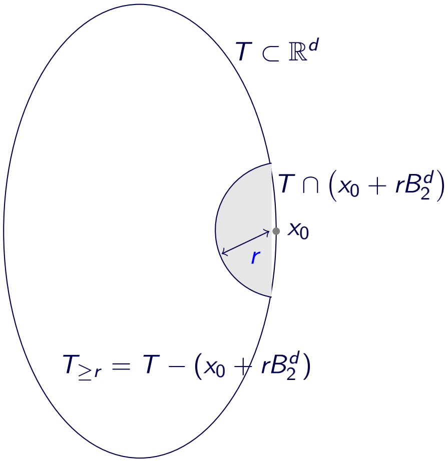

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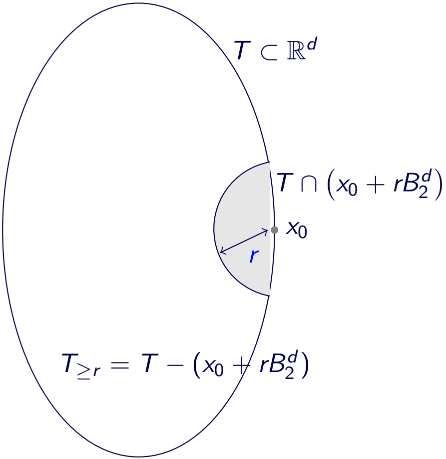

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Measure of complexity of $T_{\geq r}$?
$\ell(V) = \mathbb{E}\sup_{v\in V}\left|\sum_{j=1}^d g_j v_j\right|$: Gaussian mean width

Regularity on the complexity structure : the star-shaped assumption

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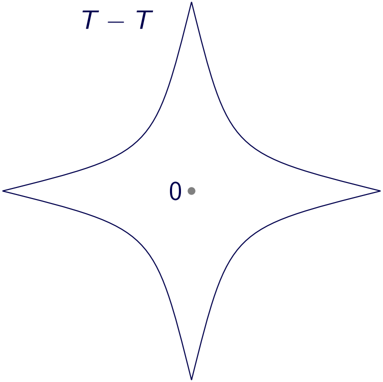

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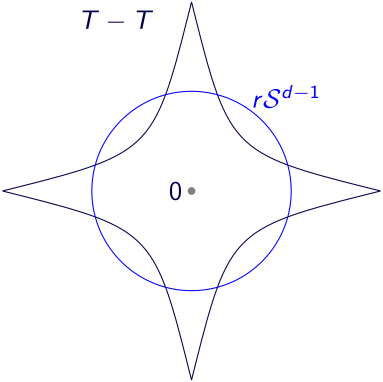

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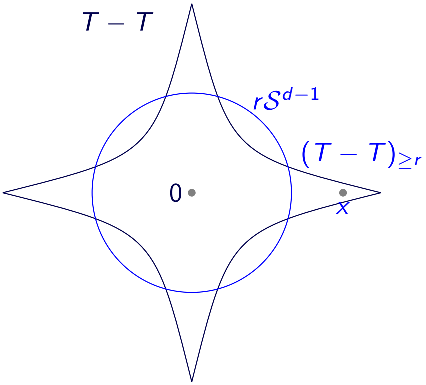

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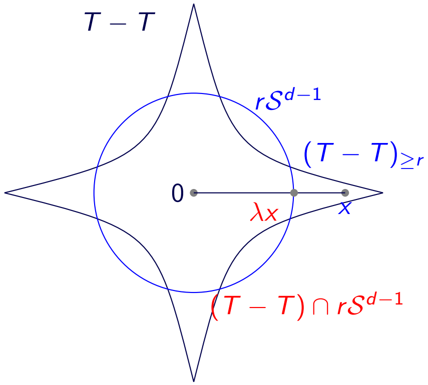

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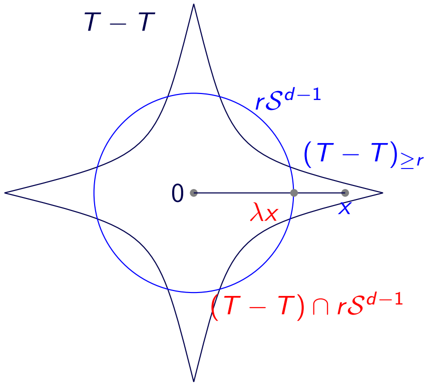
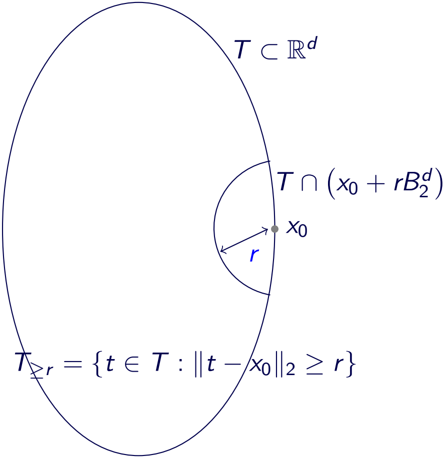


Complexity of localized sets : $(T - T) \cap r\mathcal{S}^{d-1}$
Other ways to study the complexity of $(T - T) \cap r\mathcal{S}^{d-1}$ via "peeling" cf. S. van de Geer, Cambridge University Press

Isomorphic method (Bartlett and Mendelson, PTRF06)



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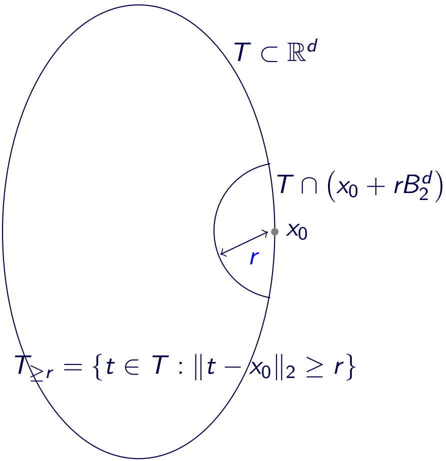

ERM : $\hat{x} \in \mathrm{argmin}_{x \in T} P_N \ell_x$,
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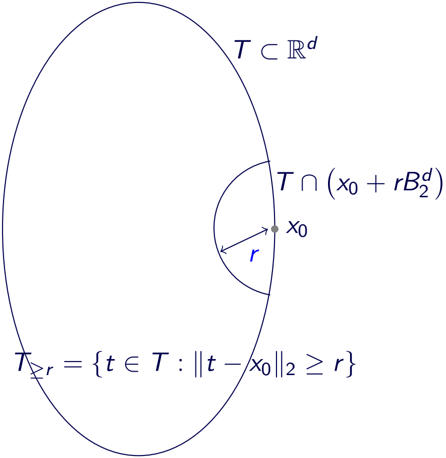

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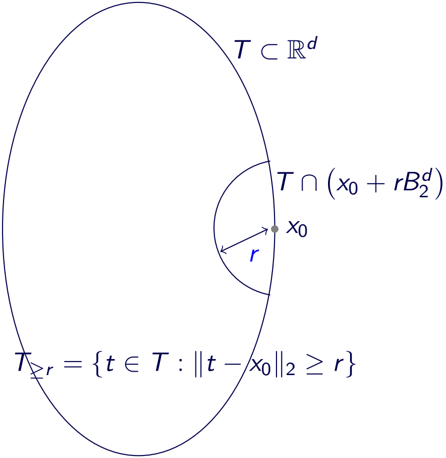

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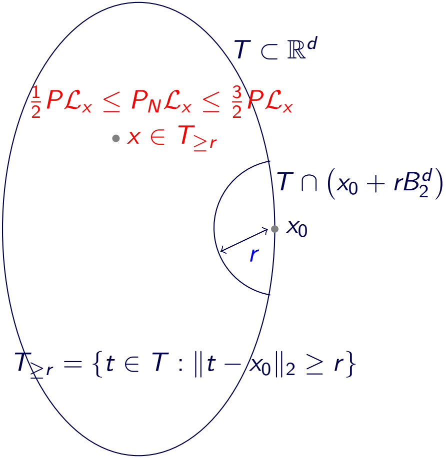

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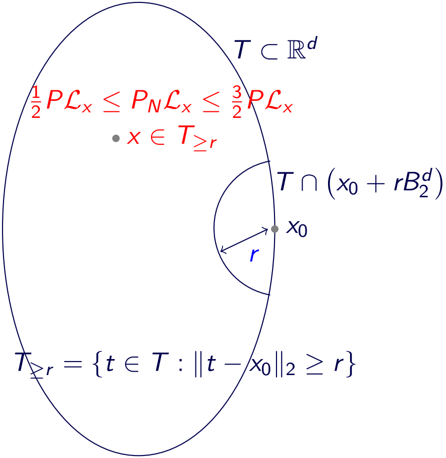

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Isomorphic property over $T_{\ge r}$ implies that $\hat{x} \notin T_{\ge r}$

$$\Longrightarrow \|\hat{x} - x_0\|_2 \le r$$

Isomorphic property over $T_{\geq r}$

We want : w.h.p. for any $x \in T_{\geq r}$,

$$\left| P_N \mathcal{L}_x - P\mathcal{L}_x \right| \leq \frac{1}{2} P\mathcal{L}_x$$

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Study of the ratio process (cf. Koltchinskii, Saint-Flour) :

$$\sup_{x \in T_{\geq r}} \left| 1 - P_N \Big(\frac{\mathcal{L}_x}{P\mathcal{L}_x} \Big) \right| \leq \frac{1}{2}$$

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Here : ratio process via decomposition of the excess risk quadratic + multiplier

$$\begin{aligned}\mathcal{L}_x(a,y) &= (\ell_x - \ell_{x_0})(a,y) = (y - \langle a,x\rangle)^2 - (y - \langle a,x_0\rangle)^2\\ &= \langle a, x - x_0\rangle^2 + 2\sigma g\langle a, x - x_0\rangle\end{aligned}$$

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$$P\mathcal{L}_x(a,y) = P\left\langle a, x - x_0 \right\rangle^2 + 2\sigma P[g\left\langle a, x - x_0 \right\rangle] = P\left\langle a, x - x_0 \right\rangle^2 = \|x - x_0\|_2^2 = r^2.$$

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2 empirical processes - 2 statistical complexities - 2 regimes

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$$\mathrm{comp}^2(T_{\geq r}) \lesssim \max\left(\sigma \frac{\ell(T \cap rB_2^d)}{\sqrt{N}}, r\frac{\ell(T \cap rB_2^d)}{\sqrt{N}} + \frac{\ell^2(T \cap rB_2^d)}{N}\right).$$

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Phase recovery

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rem. : Even when T is convex, this is not a convex optimization problem (cf. E. Candès et al. or A. d'Aspremont for linear programming algorithms in phase recovery).

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Definition

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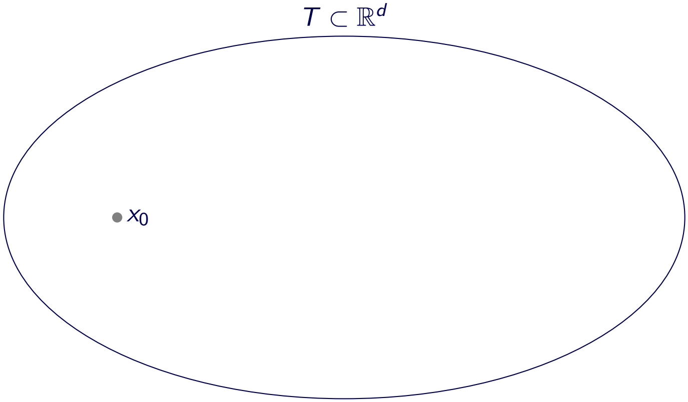
$T \subset \mathbb{R}^d$
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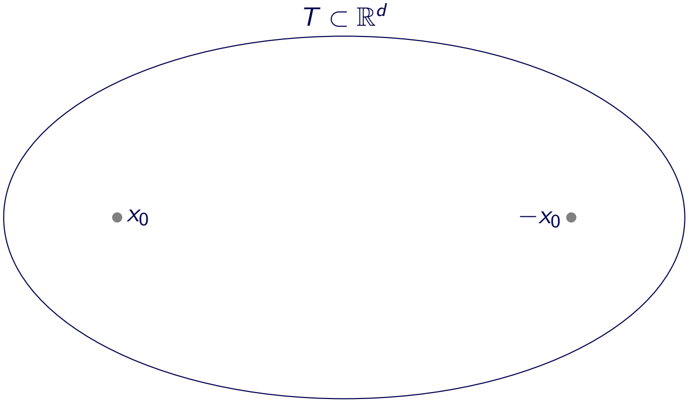
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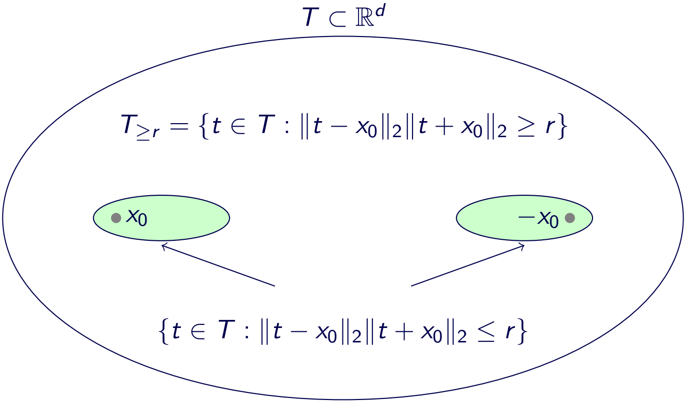
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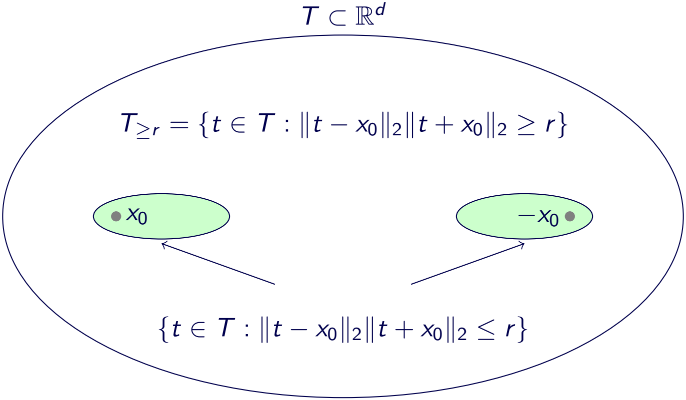

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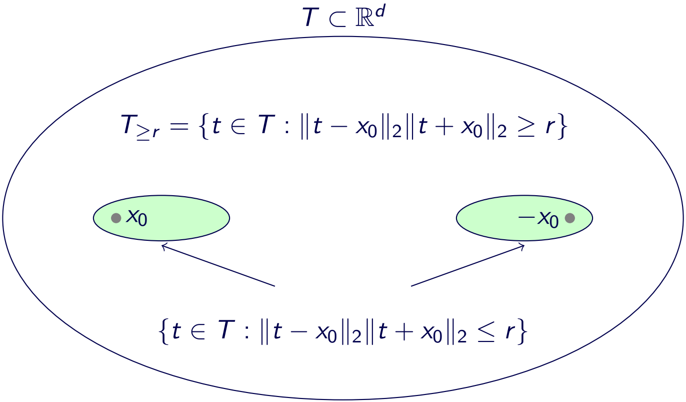

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Measure of complexity :

$$E_r = \max\left(\ell(T_{-,r}), \ell(T_{+,r})\right)$$

The "quadratic" + "multiplier" decomposition

Decomposition of the empirical excess loss :

$$P_N\mathcal{L}_x = \frac{1}{N}\sum_{i=1}^{N}\left\langle a_i, x-x_0\right\rangle^2\left\langle a_i, x+x_0\right\rangle^2 - \frac{2\sigma}{N}\sum_{i=1}^{N} g_i\left\langle a_i, x-x_0\right\rangle\left\langle a_i, x+x_0\right\rangle$$

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power 3 of a subgaussian variables ($\sim \psi_{2/3}$) - control via contraction principle : a $\sqrt{\log N}$ extra term.

Control of the "quadratic" term via empirical small ball estimate

Proposition

If $\sqrt{N} \gtrsim E_r$ then w.h.p. for any $t \in T$ such that $\|t - x_0\|_2 \|t + x_0\|_2 \geq r$, there exists $I_t \subset \{1, \dots, N\}$ such that $|I_t| \gtrsim N$

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Sharp control of the "quadratic term" as long as $\sqrt{N} \gtrsim E_r$: "fixed point equation" for r when the noise is small.

Control of the "multiplier" process

w.h.p. for any $t \in T$ such that $\|t - x_0\|_2 \|t + x_0\|_2 \geq r$,

$$\left| \frac{1}{N} \sum_{i=1}^{N} \sigma g_i \langle a_i, t - x_0 \rangle \langle a_i, t + x_0 \rangle \right| \lesssim \sigma \sqrt{\log N} \frac{E_r}{\sqrt{N}} \|t - x_0\|_2 \|t + x_0\|.$$

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Following these two estimates : if $\sqrt{N} \gtrsim E_r$ and if

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then for all $t \in T_{\geq r}$, $P_N \mathcal{L}_t > 0$ but $P_N \mathcal{L}_{\hat{x}} \leq 0$ therefore, $\hat{x} \notin T_{\geq r}$:

$$\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2 \leq r.$$

Two fixed points

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Nevertheless, it is easier to understand a result like

$$\min\left(\|\hat{x} - x_0\|_2, \|\hat{x} + x_0\|_2\right) \leq rate.$$

Relation between $\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2$ and $\min\left(\|\hat{x} - x_0\|_2, \|\hat{x} + x_0\|_2\right)$

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$\Rightarrow$ 2 regimes for (the localization and thus) the complexity term E_r depending on $\|x_0\|_2$.

$$r_N^* = \inf\left(r > 0 : \ell(2T \cap rB_2^d) \lesssim r\sqrt{N}\right)$$

$$s_N^* = \inf\left(s > 0 : \ell(2T \cap sB_2^d) \lesssim \frac{\|x_0\|_2}{\sigma\sqrt{\log N}} s^2\sqrt{N}\right)$$

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where

$rate$	$\sigma\sqrt{\log N} \leq \|x_0\|_2 r_N^*$	$\sigma\sqrt{\log N} \geq \|x_0\|_2 r_N^*$
$\|x_0\|_2 \geq v_N^*$	r_N^*	s_N^*
$\|x_0\|_2 \leq v_N^*$	r_N^*	v_N^*

Sparse vectors

$$T = W_s = \{x \in \mathbb{R}^d : |\mathrm{supp}(x)| \le s\}$$

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1. Gaussian complexity of localized sets :

$$\ell(W_s \cap rB_2^n) \sim r\sqrt{s \log\left(ed/s\right)}.$$

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Sudakov inequality is sharp : $r \log^{1/2} N\left(W_s \cap 2rB_2^d, rB_2^d\right) \sim \ell(W_s \cap rB_2^n)$

$$\Downarrow$$

ERM is minimax in linear regression and phase recovery (up to $\sqrt{\log N}$) in the noisy setup.

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Rates of convergence over W_s in linear regression and phase recovery

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$$\min\left(\|\hat{x} - x_0\|_2, \|\hat{x} + x_0\|_2\right) \leq v_N^*\left(\sigma^{-1}\right) \sim \left[\sigma \sqrt{\frac{s \log\left(ed/s\right)}{N}}\right]^{1/2}$$

The unit B_1^d-ball

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$$\ell(B_1^d \cap rB_2^d) \sim \begin{cases} \sqrt{\log\left(edr^2\right)} & \text{if } d^2 r \geq 1 \\ r\sqrt{d} & \text{otherwise} \end{cases}$$

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2. Sudakov complexity of localized sets is sharp : for any $r > 0$,

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$$\Downarrow$$

ERM is minimax in linear regression and phase recovery (up to $\sqrt{\log N}$) in the noisy case (and also in the noise free case).

Computing the three fixed points

$$r_N^*(Q) \begin{cases} \sim \left(\frac{1}{Q^2 N}\log\left(\frac{n}{Q^2 N}\right)\right)^{1/2} & \text{if } n \geq C_0 Q^2 N \\ \lesssim \frac{1}{N} & \text{if } C_1 Q^2 N \leq n \leq C_0 Q^2 N \\ = 0 & \text{if } n \leq C_1 Q^2 N. \end{cases}$$

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v_N^*(\zeta) \sim \begin{cases} \left(\frac{1}{\zeta^2 N} \log\left(\frac{n^3}{\zeta^2 N} \right) \right)^{1/6} & \text{if } n \geq \zeta^{2/3} N^{1/3} \\ \left(\frac{n}{\zeta^2 N} \right)^{1/4} & \text{if } n \leq \zeta^{2/3} N^{1/3}. \end{cases}
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Rates of convergence over B_1^d when $\sigma = 0$

1. when $d \gtrsim N$ then
   - in Linear regression : w.h.p.

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   - in Phase recovery : w.h.p. $\hat{x} = x_0$ or $\hat{x} = -x_0$.

Thanks for your attention