Empirical risk minimization in linear regression and phase recovery

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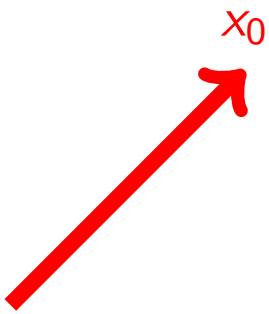
12th November 2013 - Göttingen

joint works with Shahar Mendelson

Two frameworks : linear regression and phase recovery



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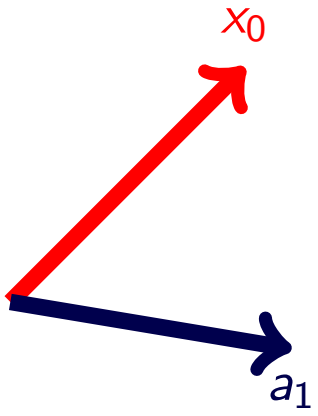

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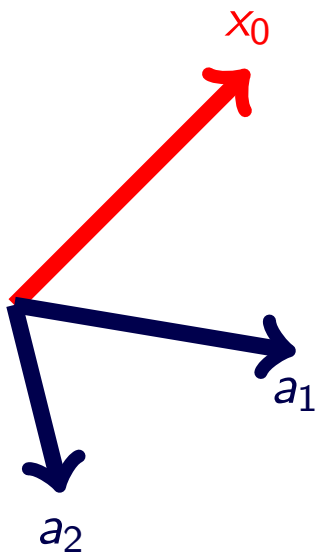

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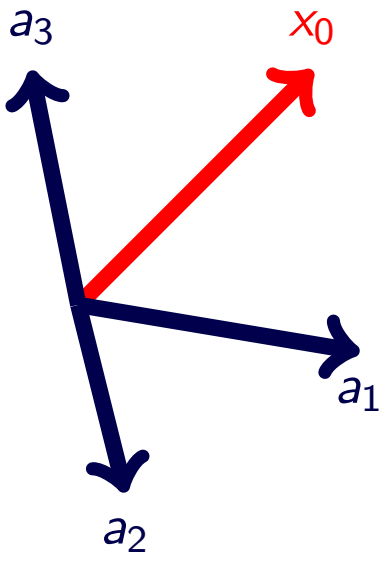

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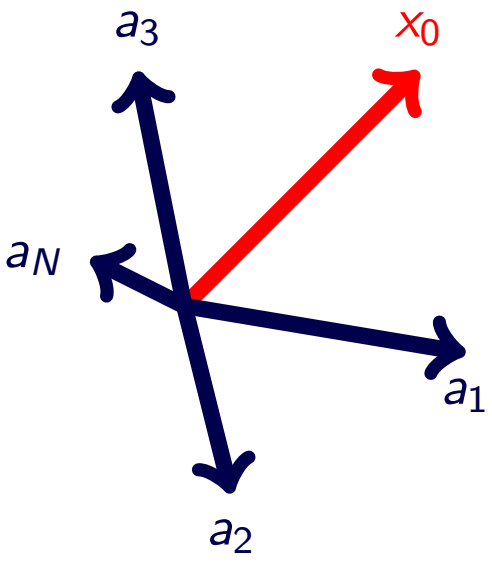
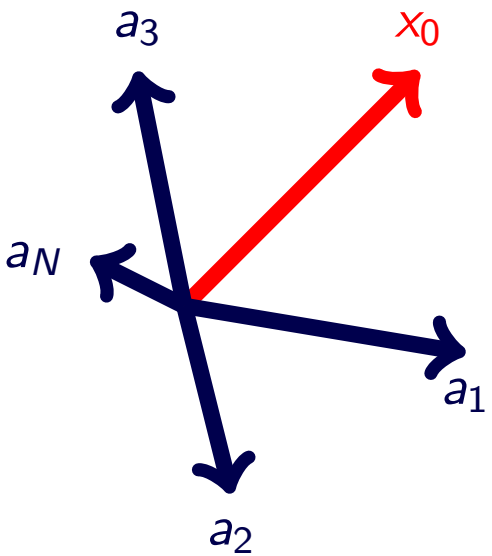


Two frameworks : linear regression and phase recovery

$T \subset \mathbb{R}^d$ and $x_0 \in T$



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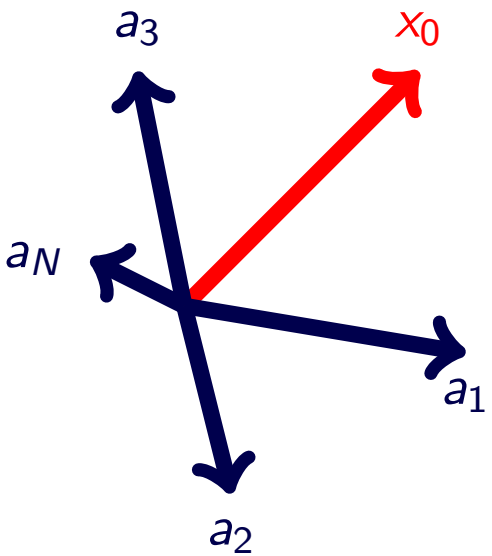

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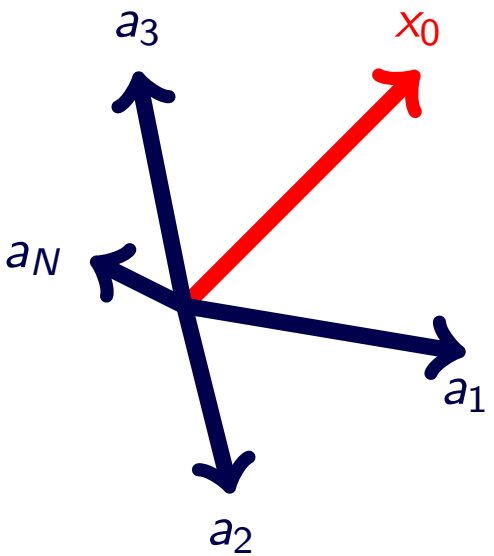

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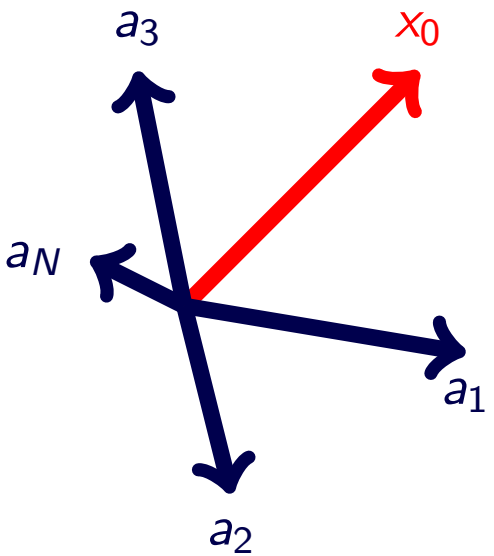

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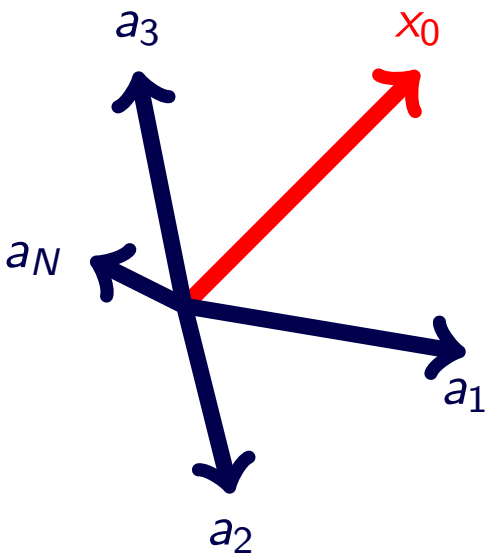

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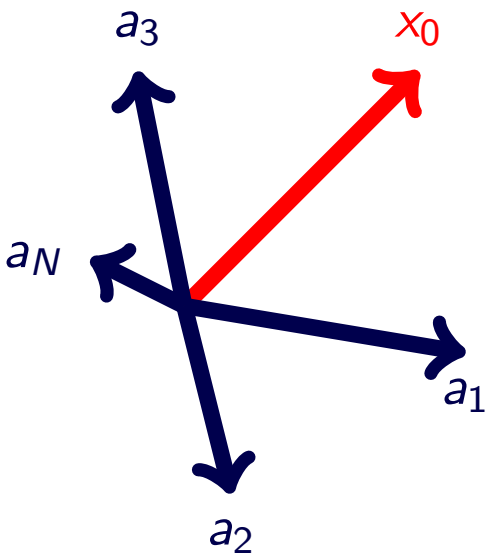

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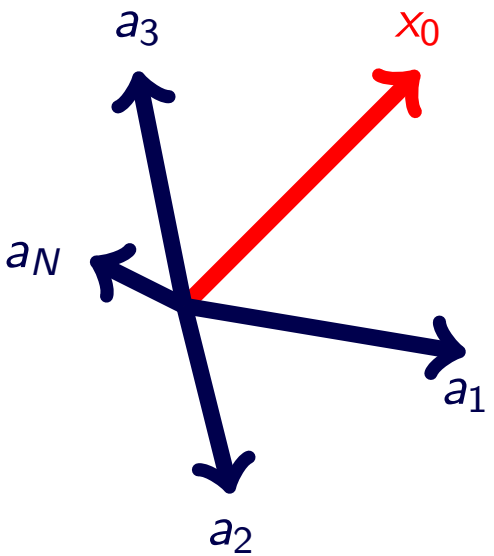

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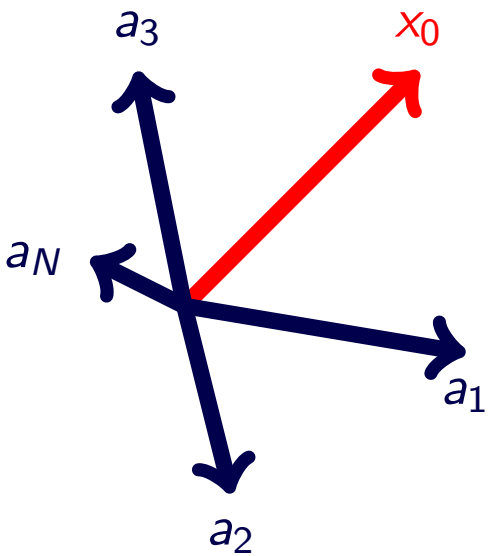

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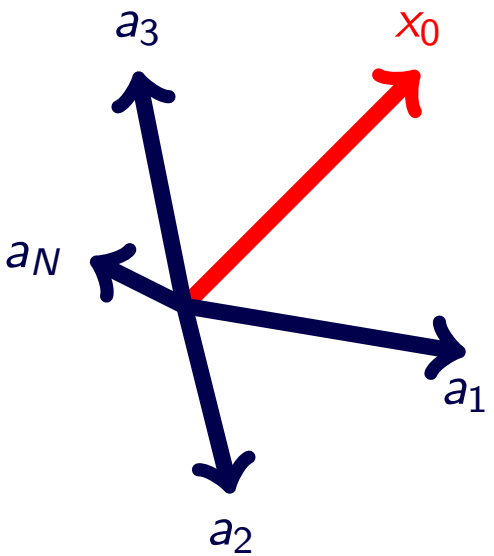

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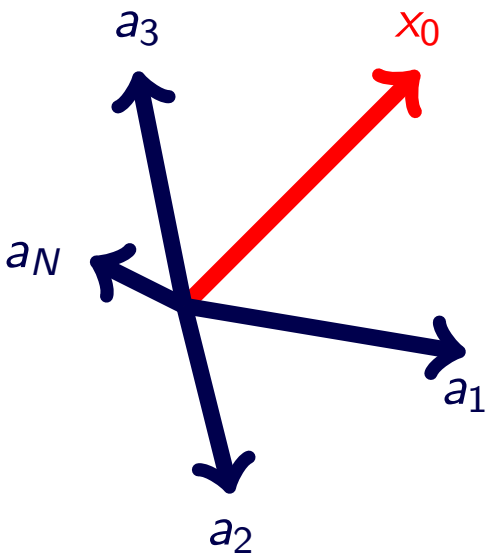

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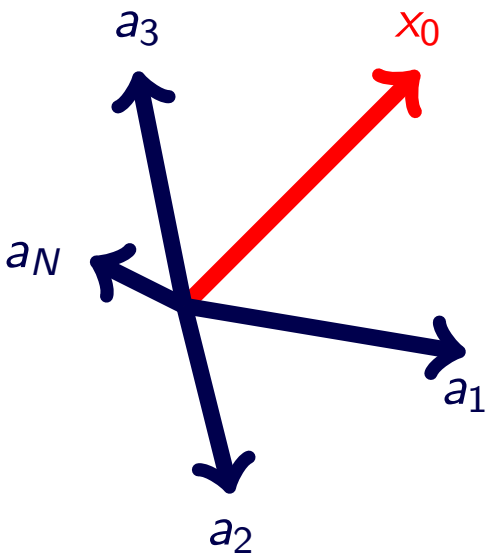

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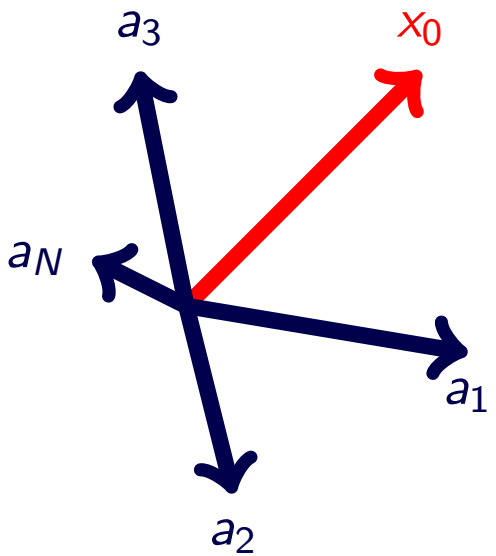

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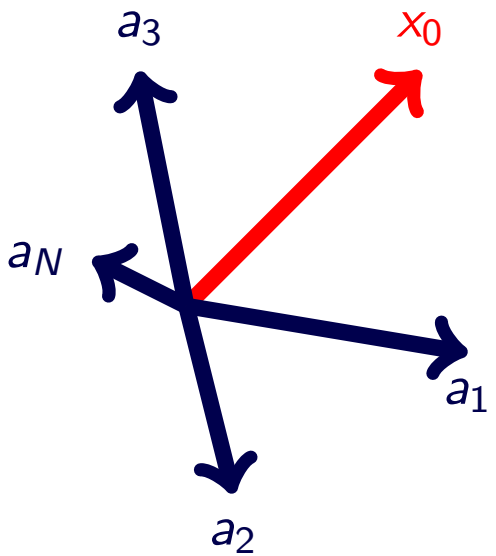

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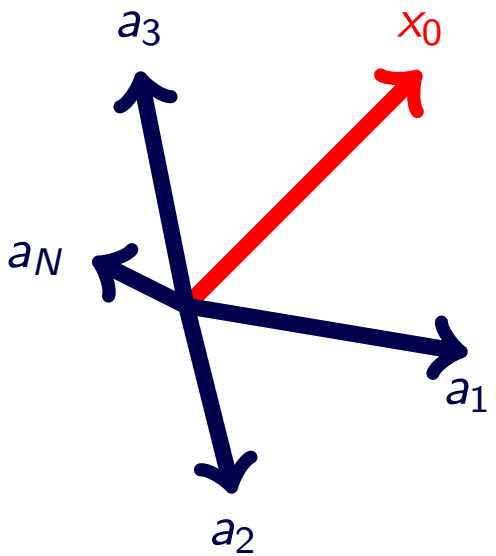

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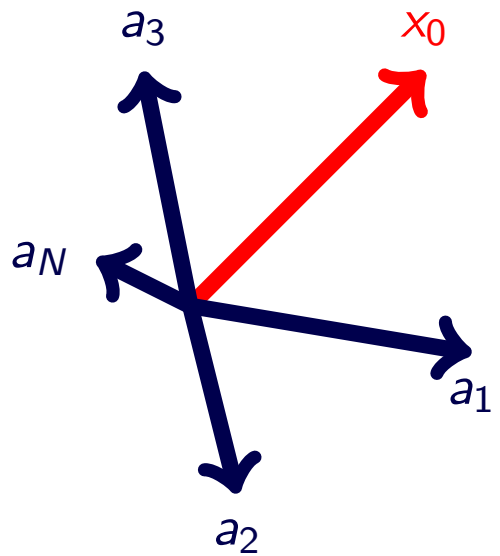

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ex. : Gaussian measurements, Rademacher measurements.

Linear regression

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Ordinary least square estimator - Maximum likelihood estimator

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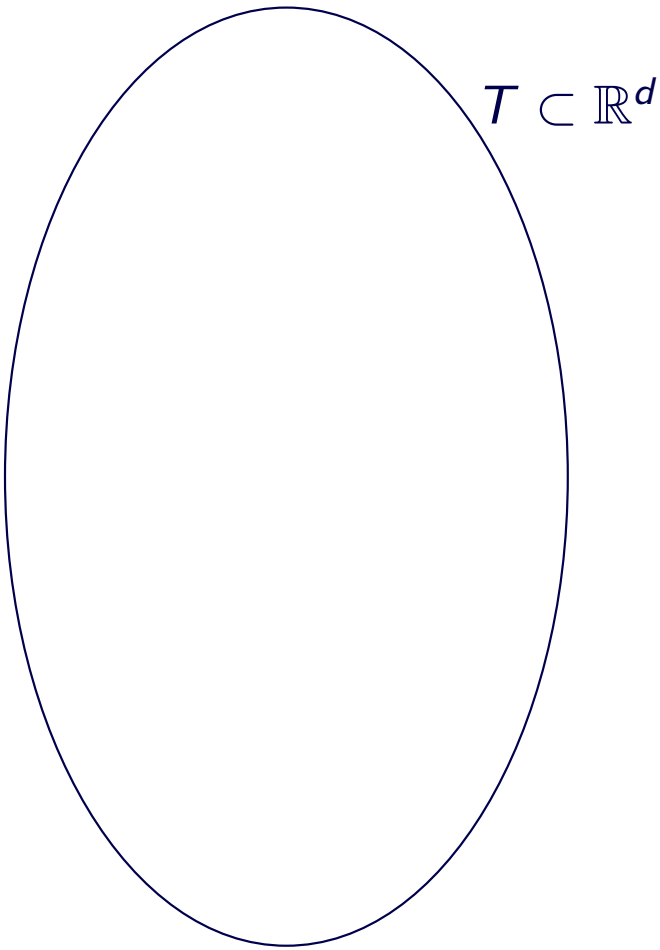

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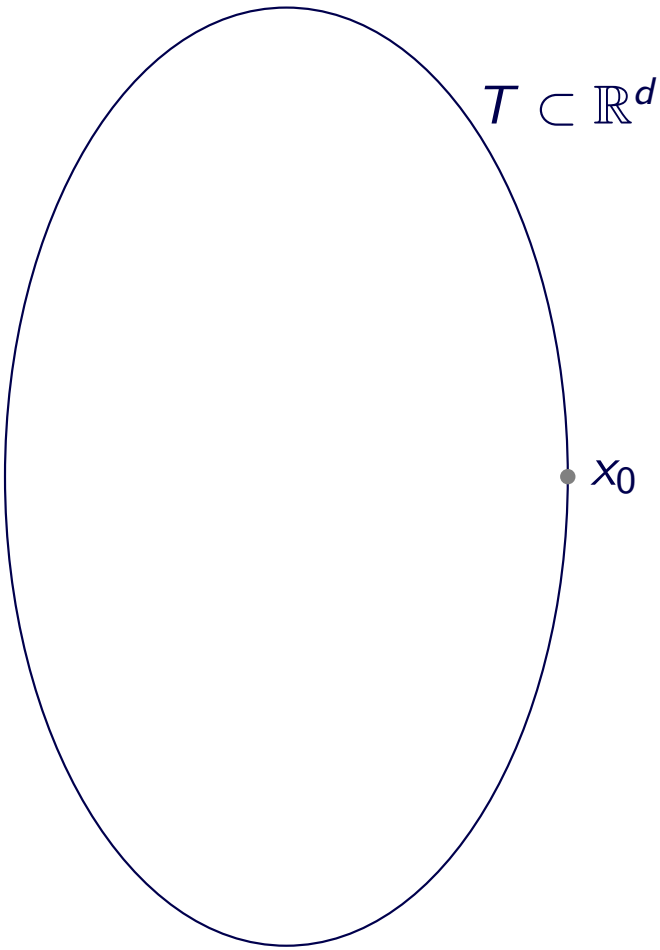

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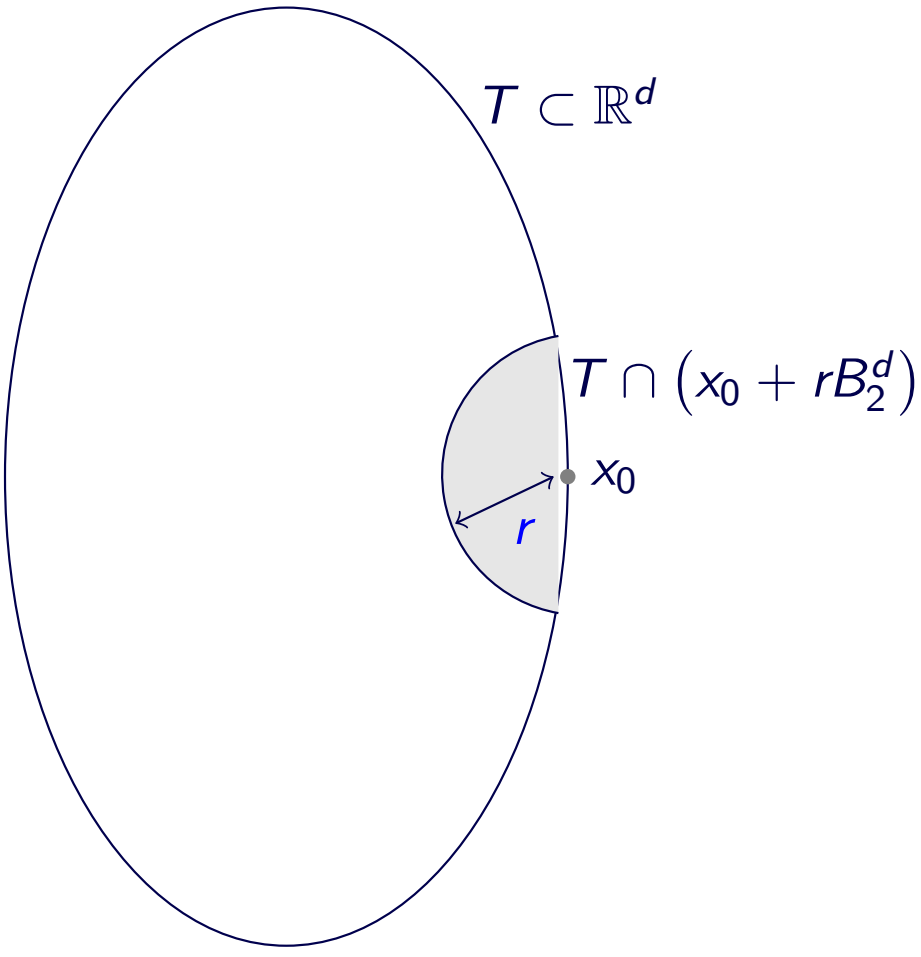

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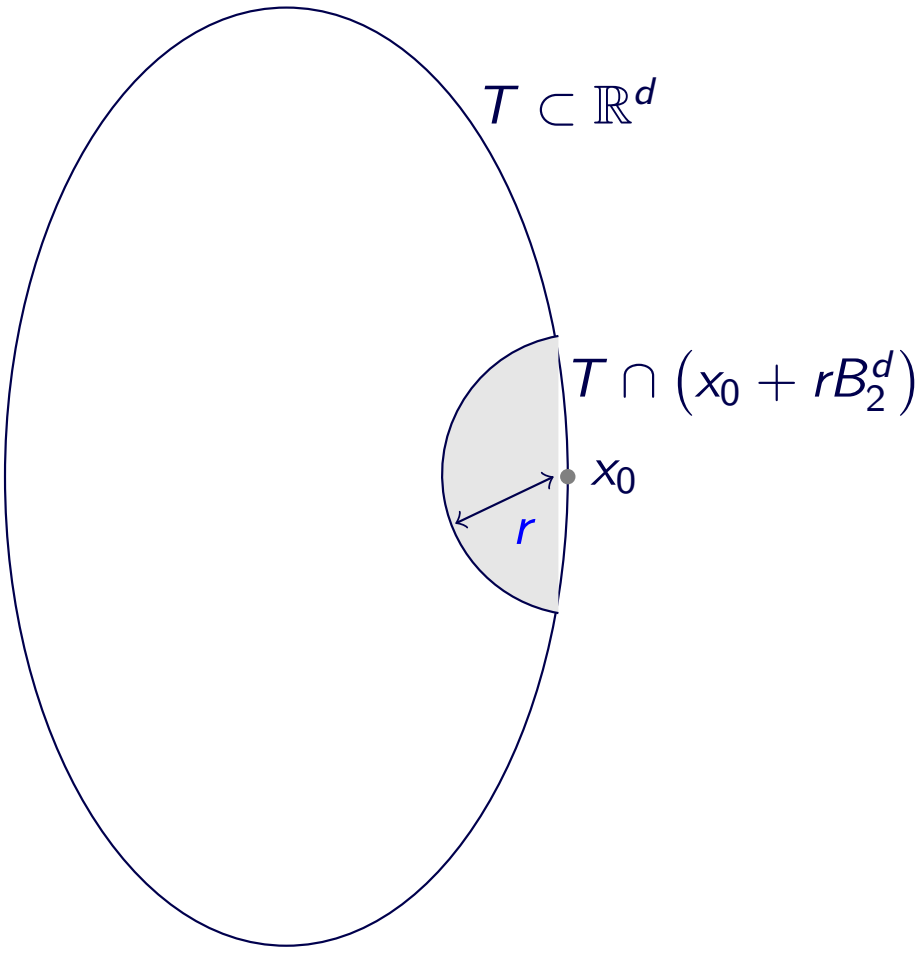

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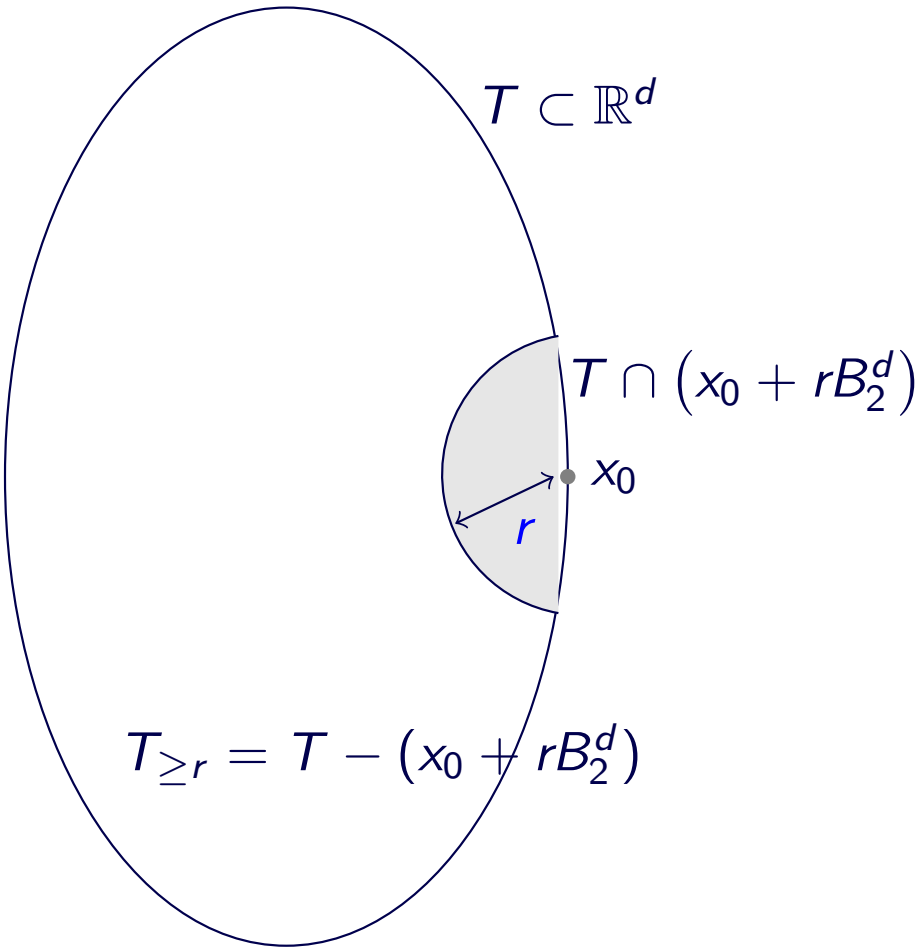

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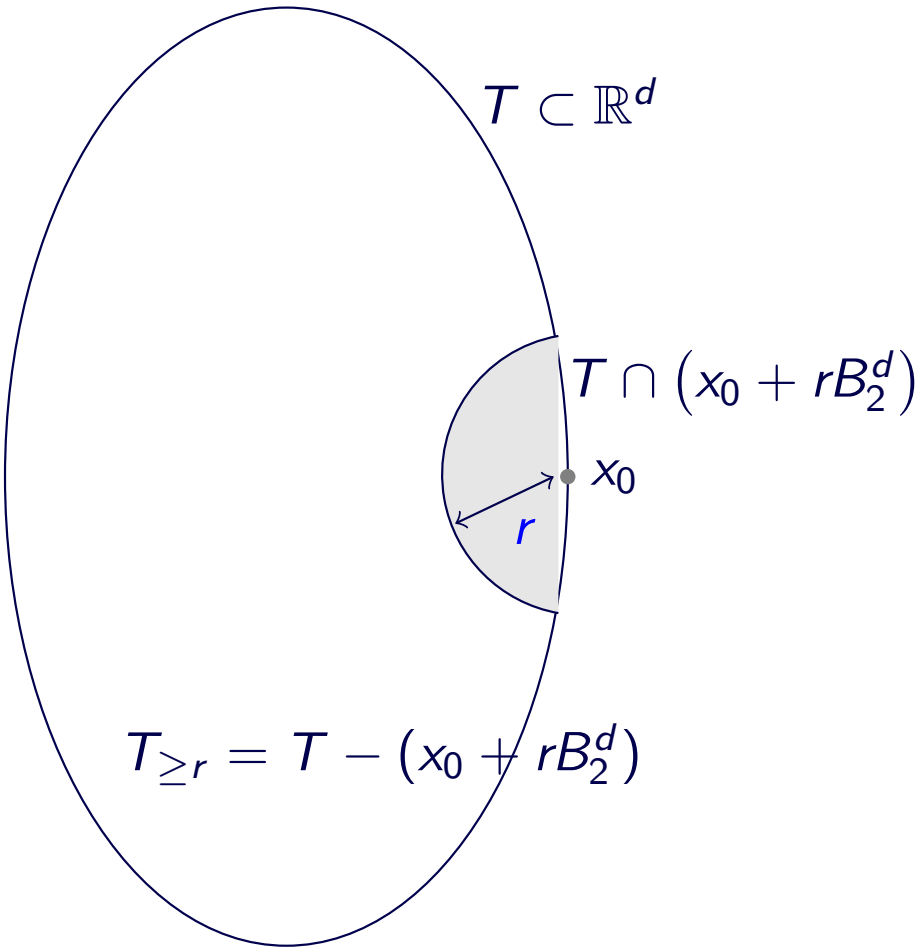

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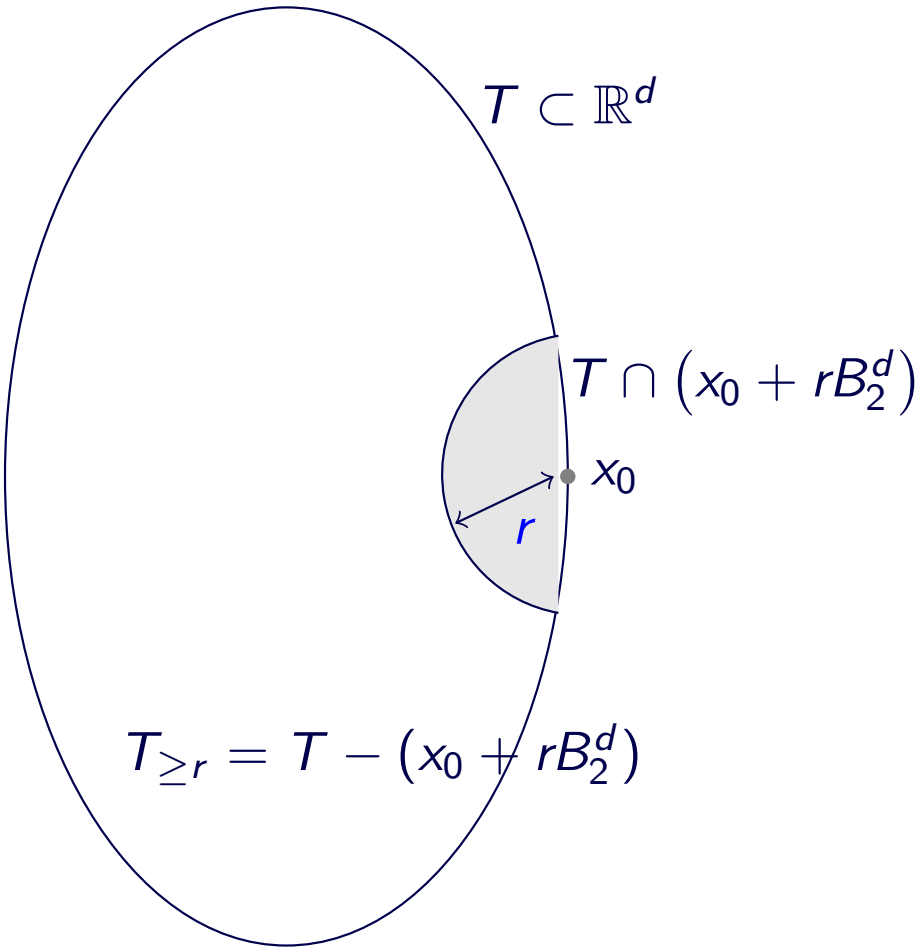

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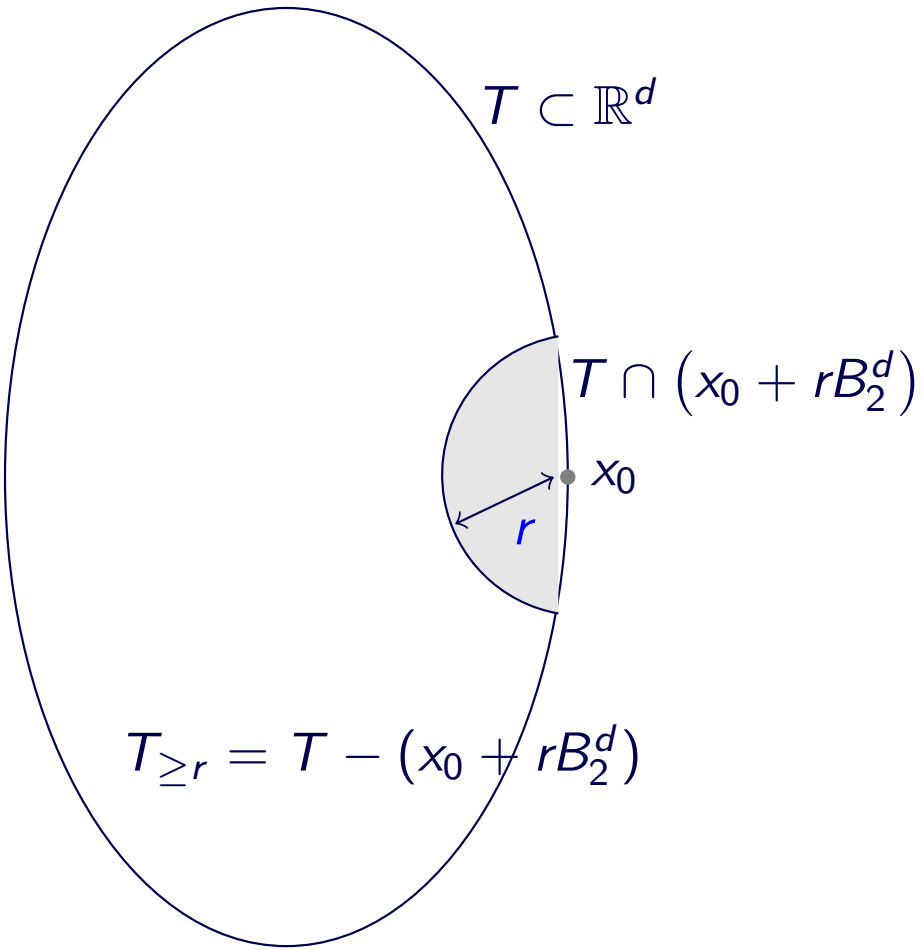

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Measure of complexity of $T_{\geq r}$?
$\ell(V) = \mathbb{E} \sup_{v \in V} \left| \sum_{j=1}^d g_j v_j \right|$: Gaussian mean width

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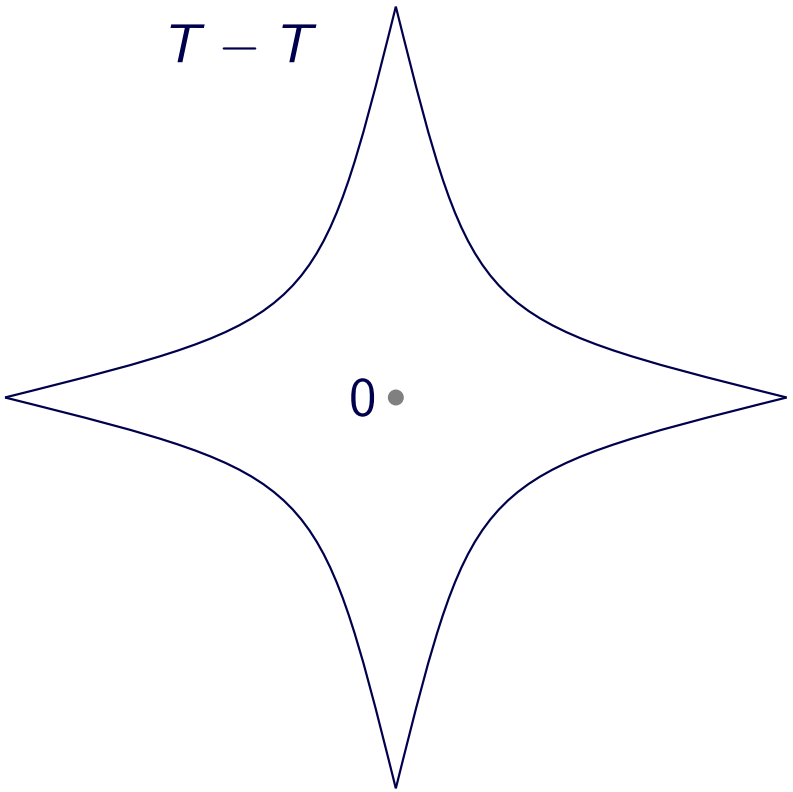

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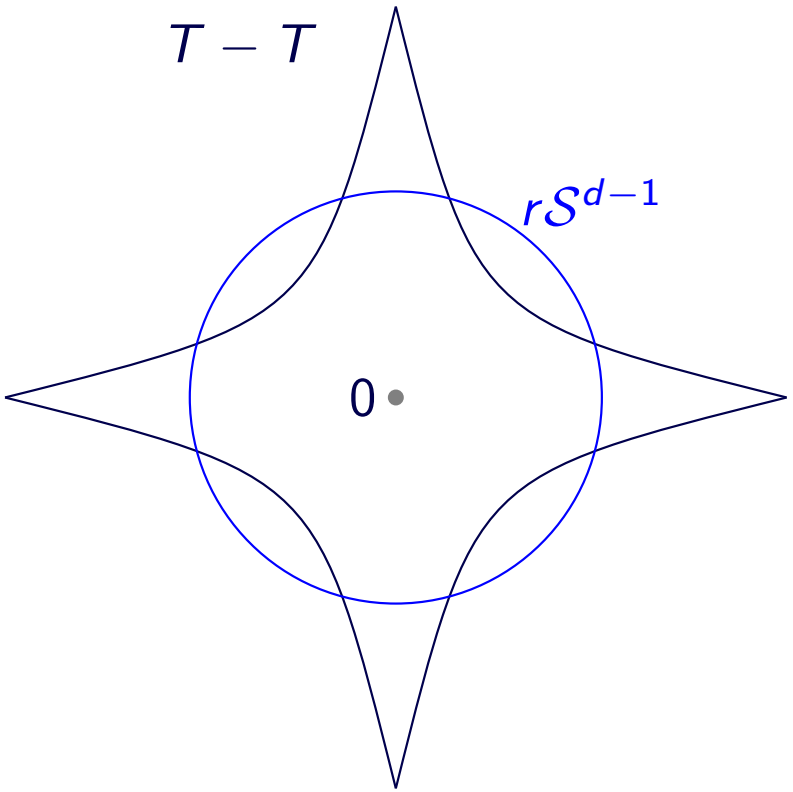

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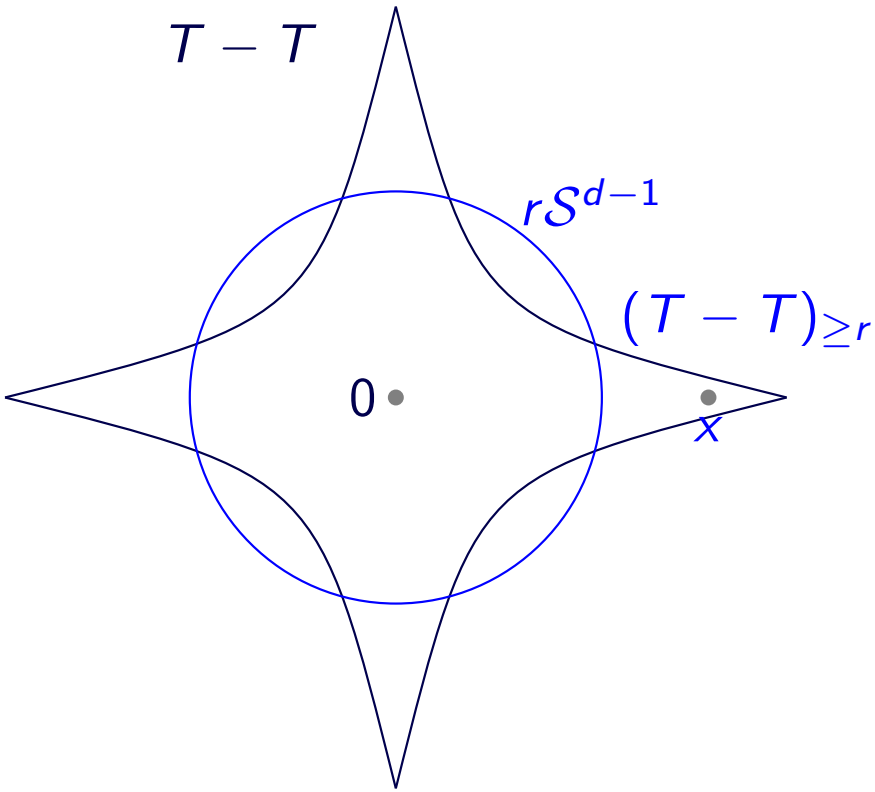

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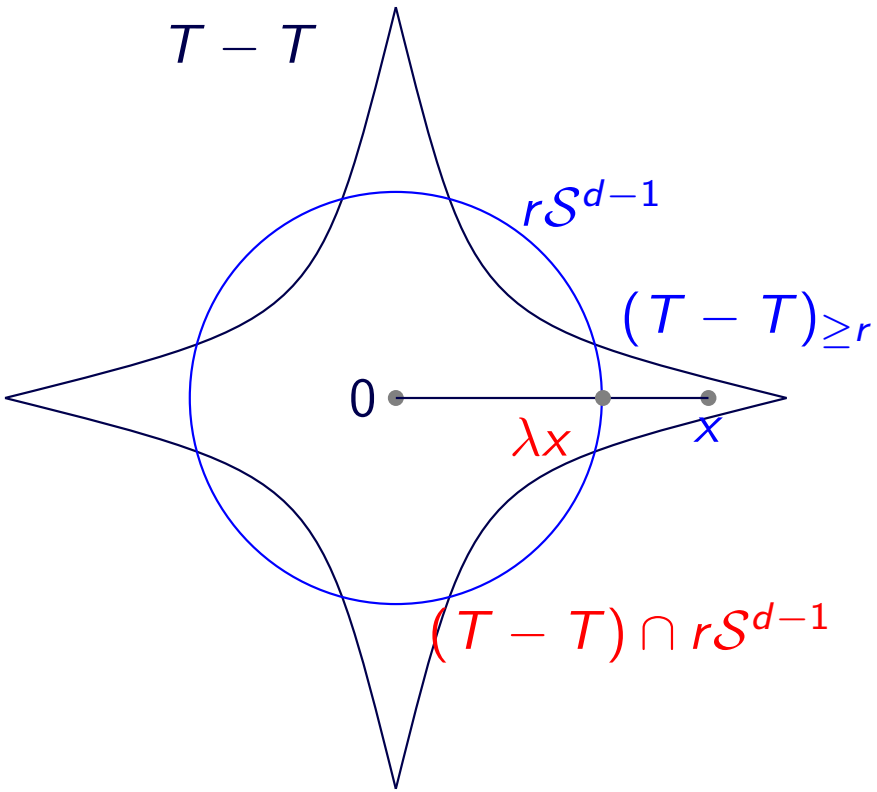

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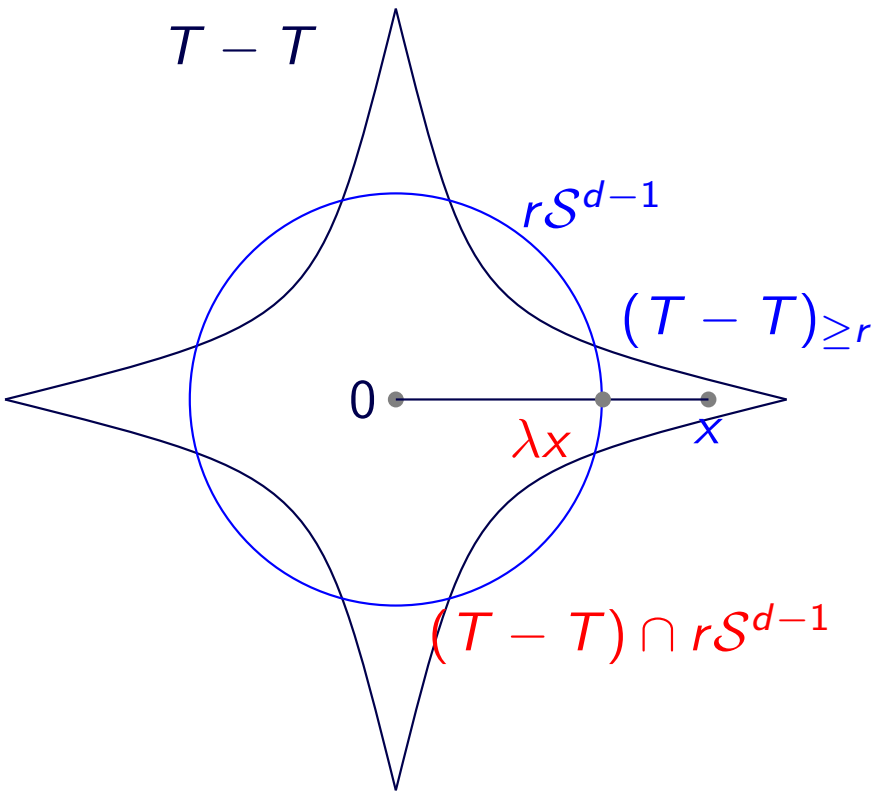
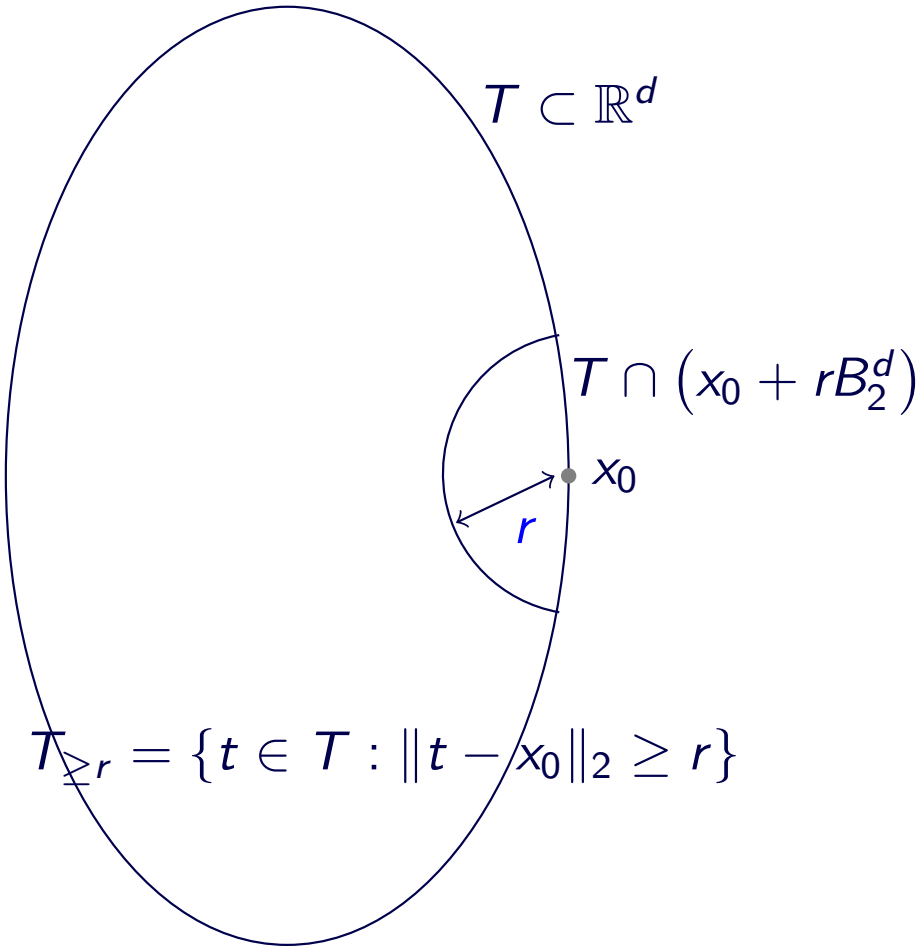


Complexity of localized sets : $(T - T) \cap r\mathcal{S}^{d-1}$
Other ways to study the complexity of $(T - T) \cap r\mathcal{S}^{d-1}$ via "peeling" cf.
S. van de Geer, Cambridge University Press

Isomorphic method (Bartlett and Mendelson, PTRF06)



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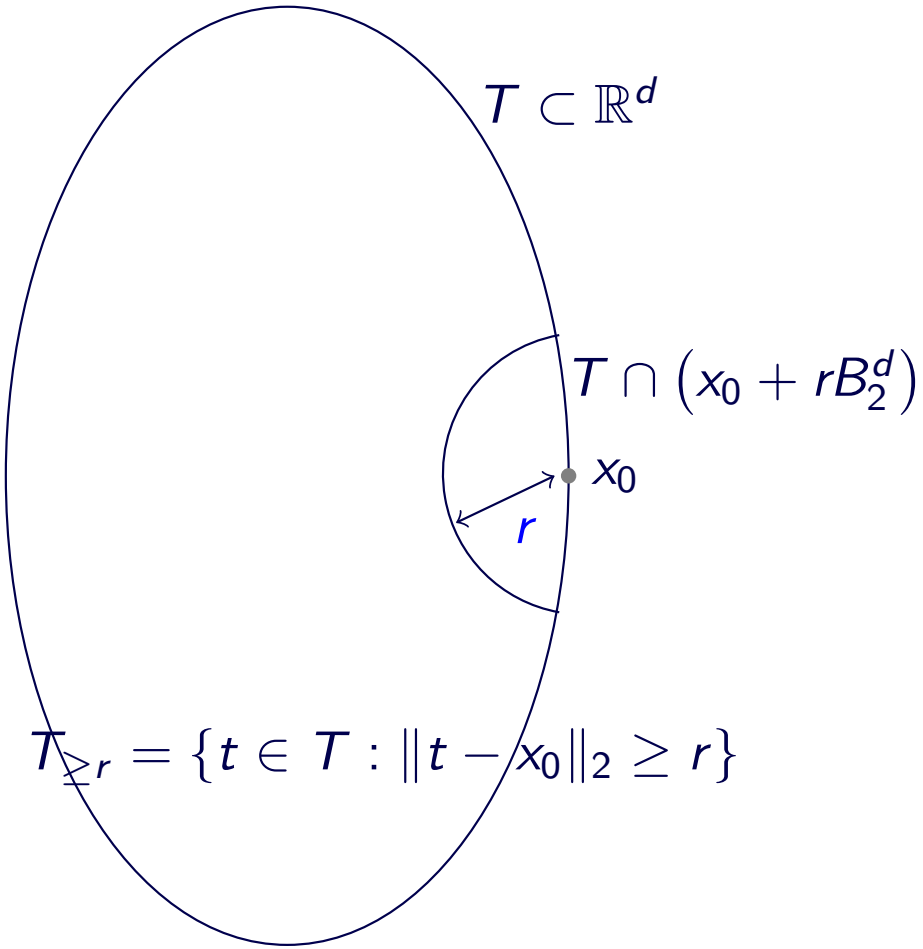

ERM : $\hat{x} \in \operatorname{argmin}_{x \in T} P_N \ell_x$,
loss function :
$\ell_x(a, y) = \left(y - \langle a, x \rangle\right)^2$

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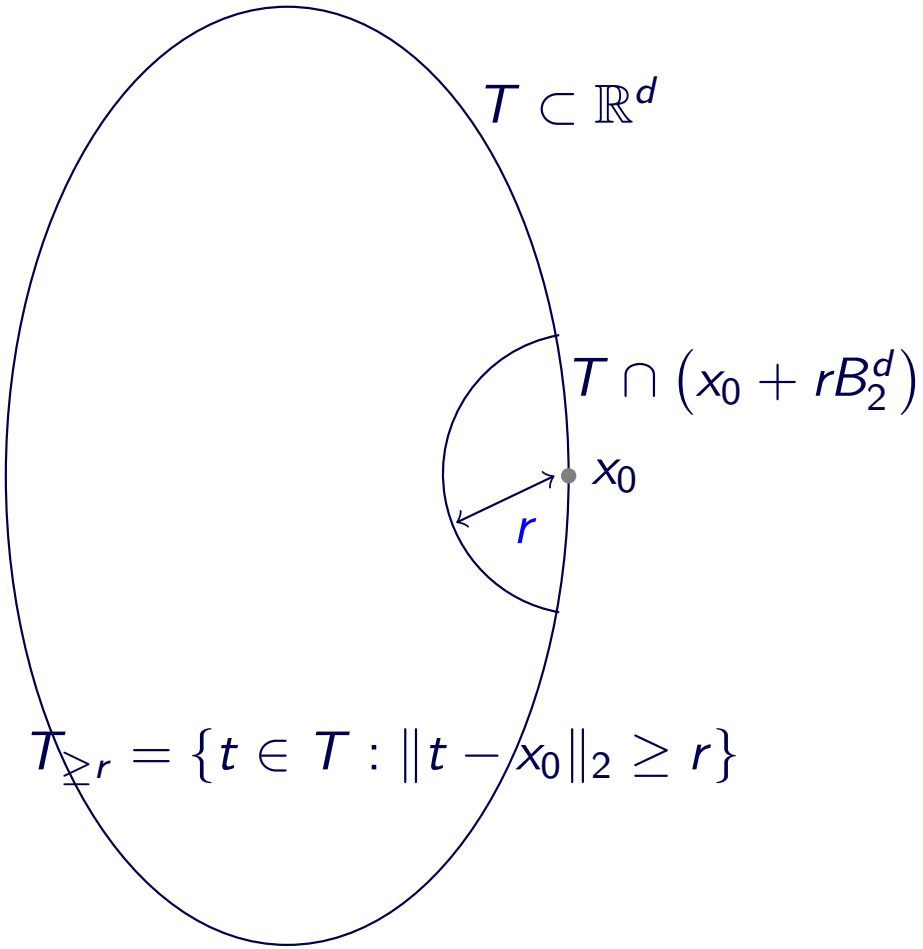

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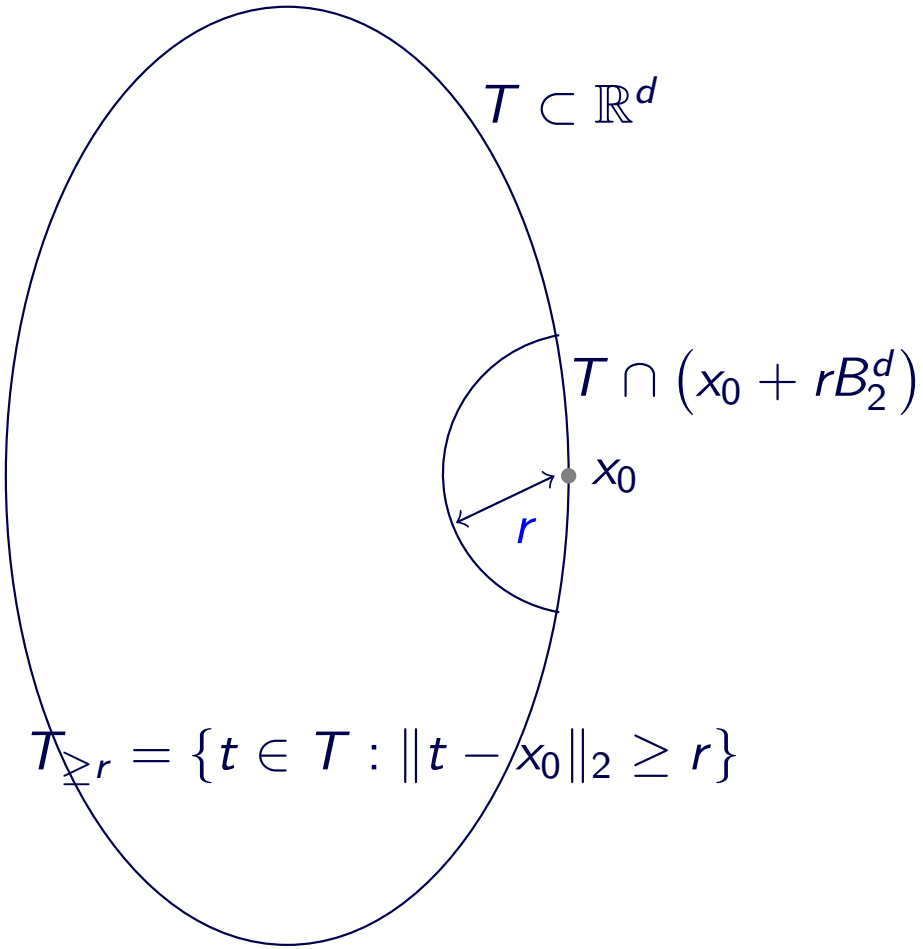

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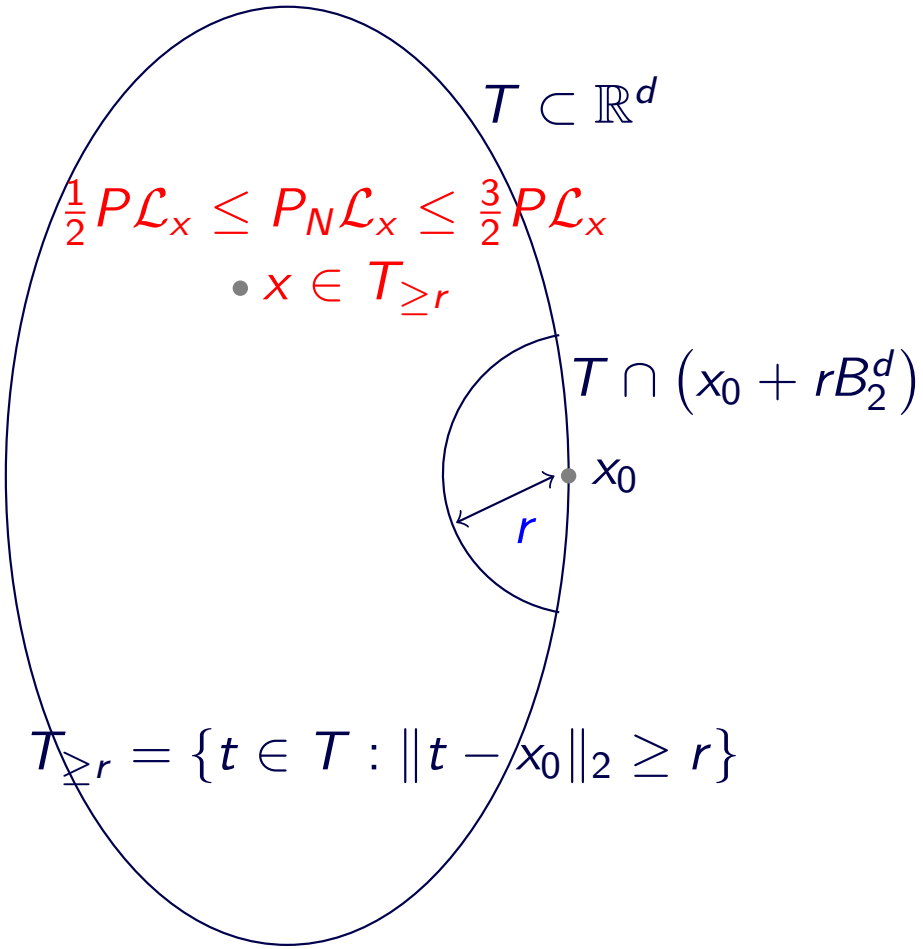

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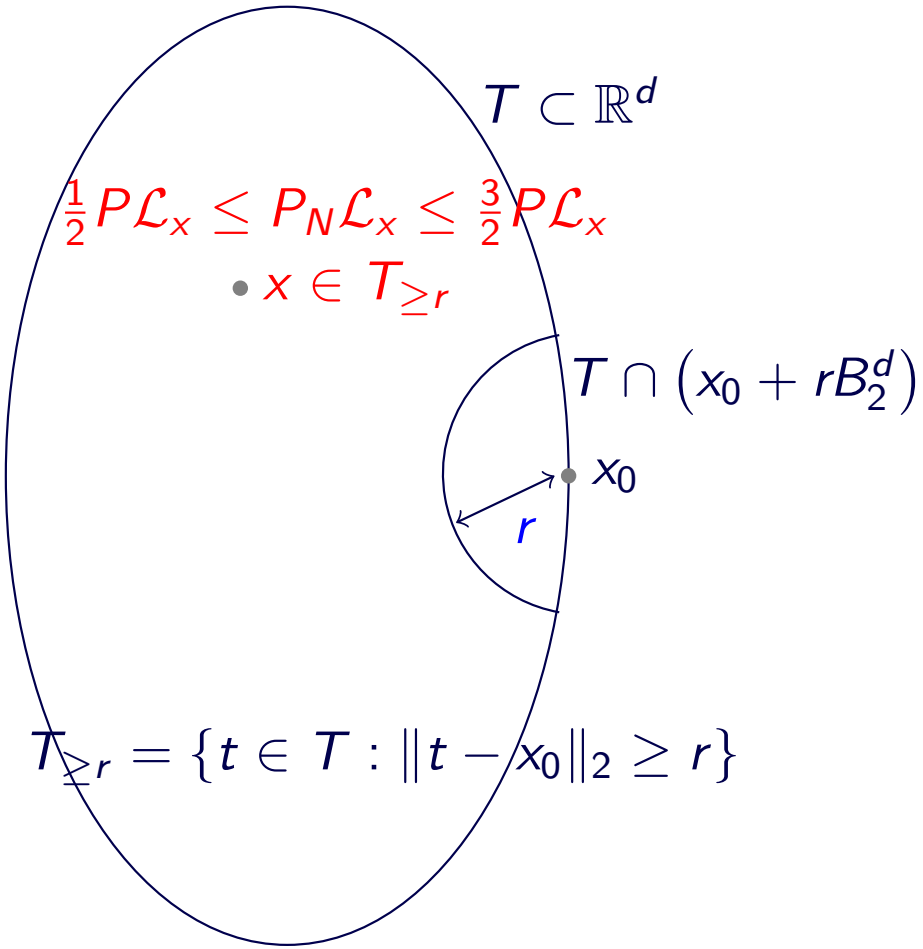

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Isomorphic property over $T_{\ge r}$ implies that $\hat{x} \notin T_{\ge r}$

$$\Longrightarrow \|\hat{x} - x_0\|_2 \le r$$

Isomorphic property over $T_{\geq r}$

We want : w.h.p. for any $x \in T_{\geq r}$,

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Here : ratio process via decomposition of the excess risk quadratic + multiplier

$$\begin{aligned}\mathcal{L}_x(a,y) = \left(\ell_x - \ell_{x_0}\right)(a,y) &= (y - \langle a, x\rangle)^2 - (y - \langle a, x_0\rangle)^2 \\ &= \langle a, x - x_0\rangle^2 + 2\sigma g \langle a, x - x_0\rangle\end{aligned}$$

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- the noise $y_i = \langle a_i, x_0\rangle + \sigma g_i$ is a source of statistical complexity. Main source of complexity when $\sigma \gtrsim r_N^*$. This complexity is measured by the multiplier process $(P - P_N)(g\langle a, u\rangle)_{u \in (T-T)\cap r\mathcal{S}^{n-1}}$.

2 empirical processes - 2 statistical complexities - 2 regimes

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$$\sup_{v \in V} \left| \frac{1}{N} \sum_{i=1}^{N} \langle a_i, v \rangle^2 - \|v\|_2^2 \right| \lesssim \left(\mathrm{diam}(V, \ell_2^d) \frac{\ell(V)}{\sqrt{N}} + \frac{\ell^2(V)}{N} \right).$$

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Phase recovery

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rem. : Even when T is convex, this is not a convex optimization problem (cf. E. Candès et al. or A. d'Aspremont for linear programming algorithms in phase recovery).

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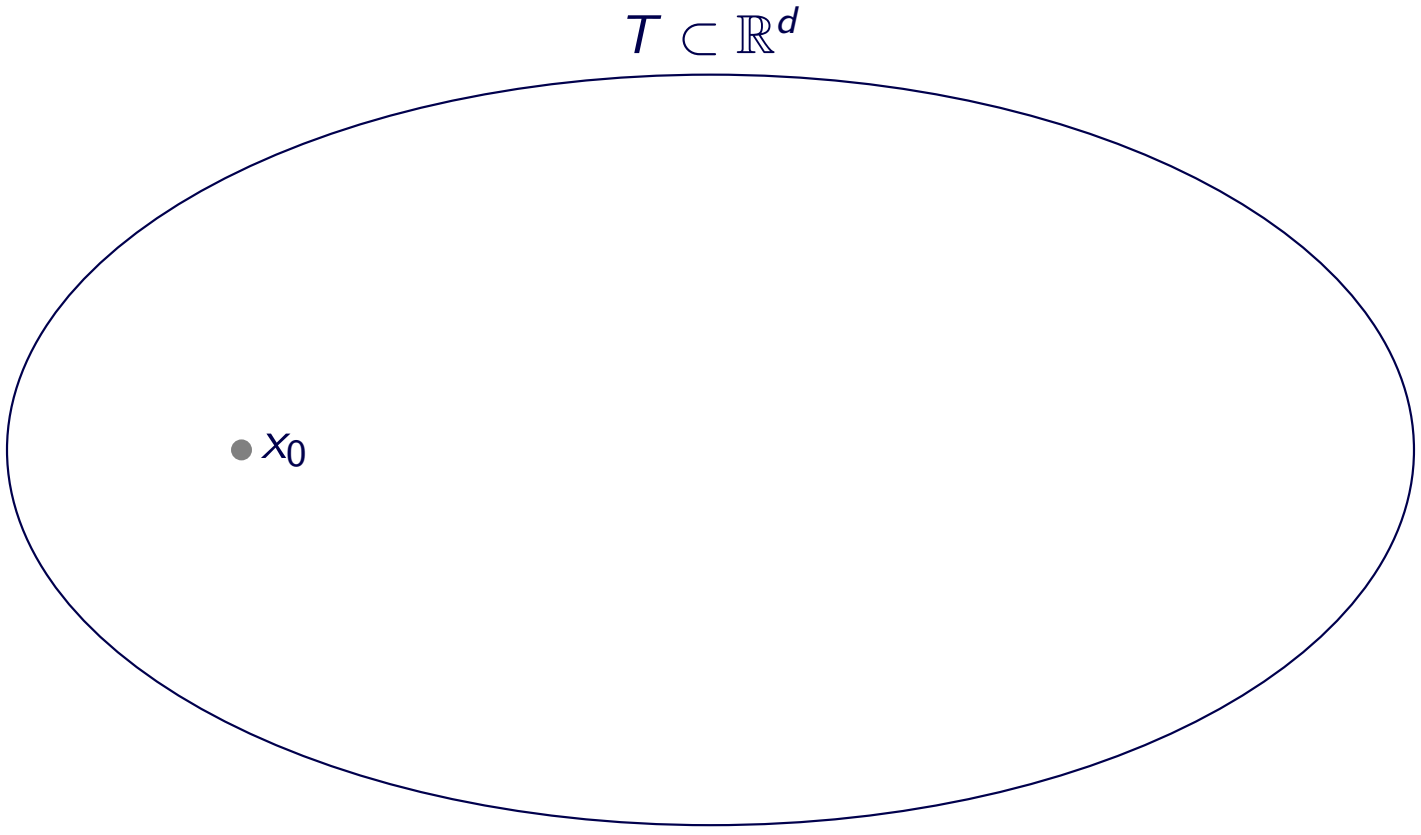
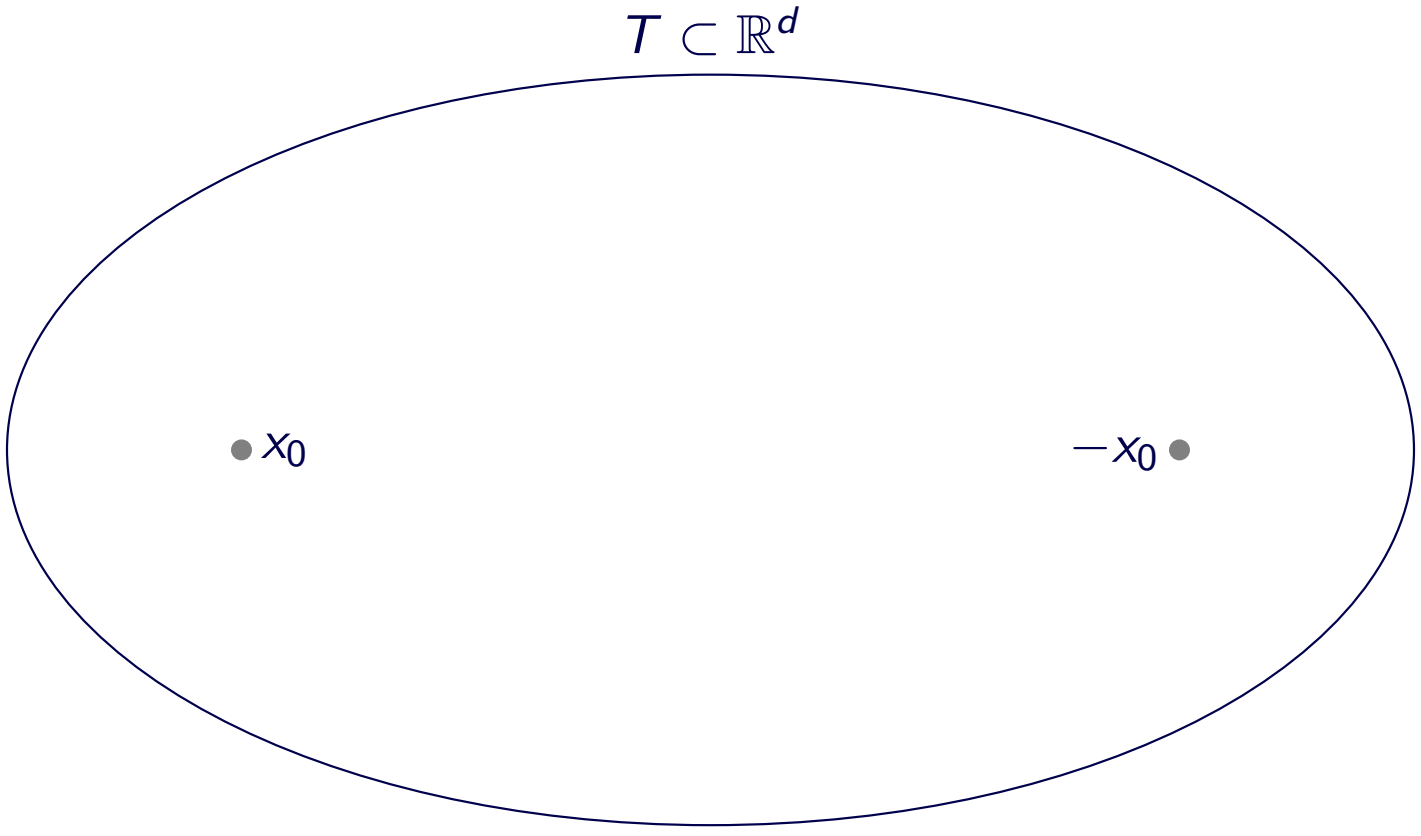
a satisfies the small ball assumption $\Rightarrow \mathbb{E}|\langle a, u\rangle\langle a, v\rangle| \geq \kappa_0 \|u\|_2 \|v\|_2$


$T \subset \mathbb{R}^d$
x_0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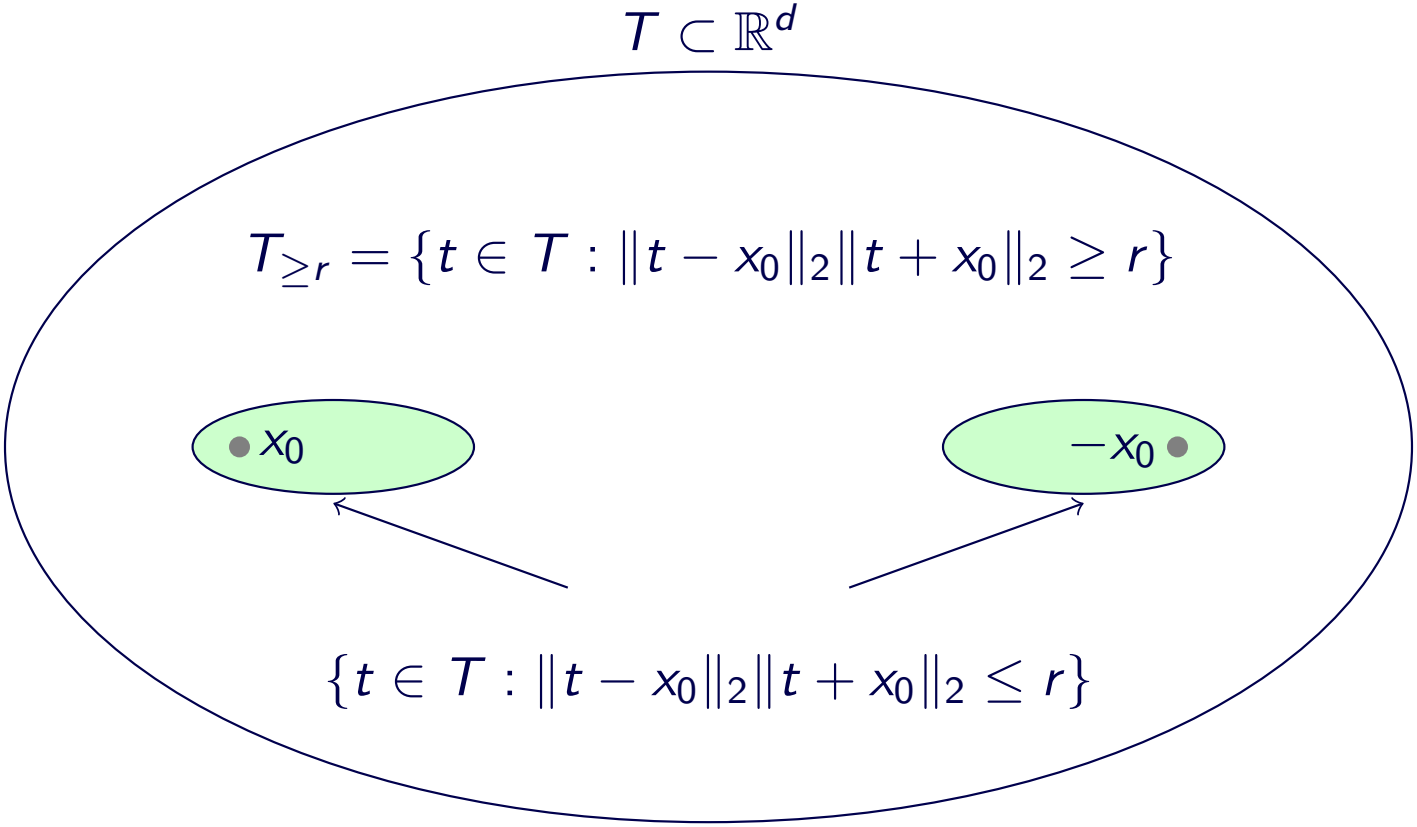
$T \subset \mathbb{R}^d$
$T_{\geq r} = \{t \in T : \|t - x_0\|_2 \|t + x_0\|_2 \geq r\}$
x_0
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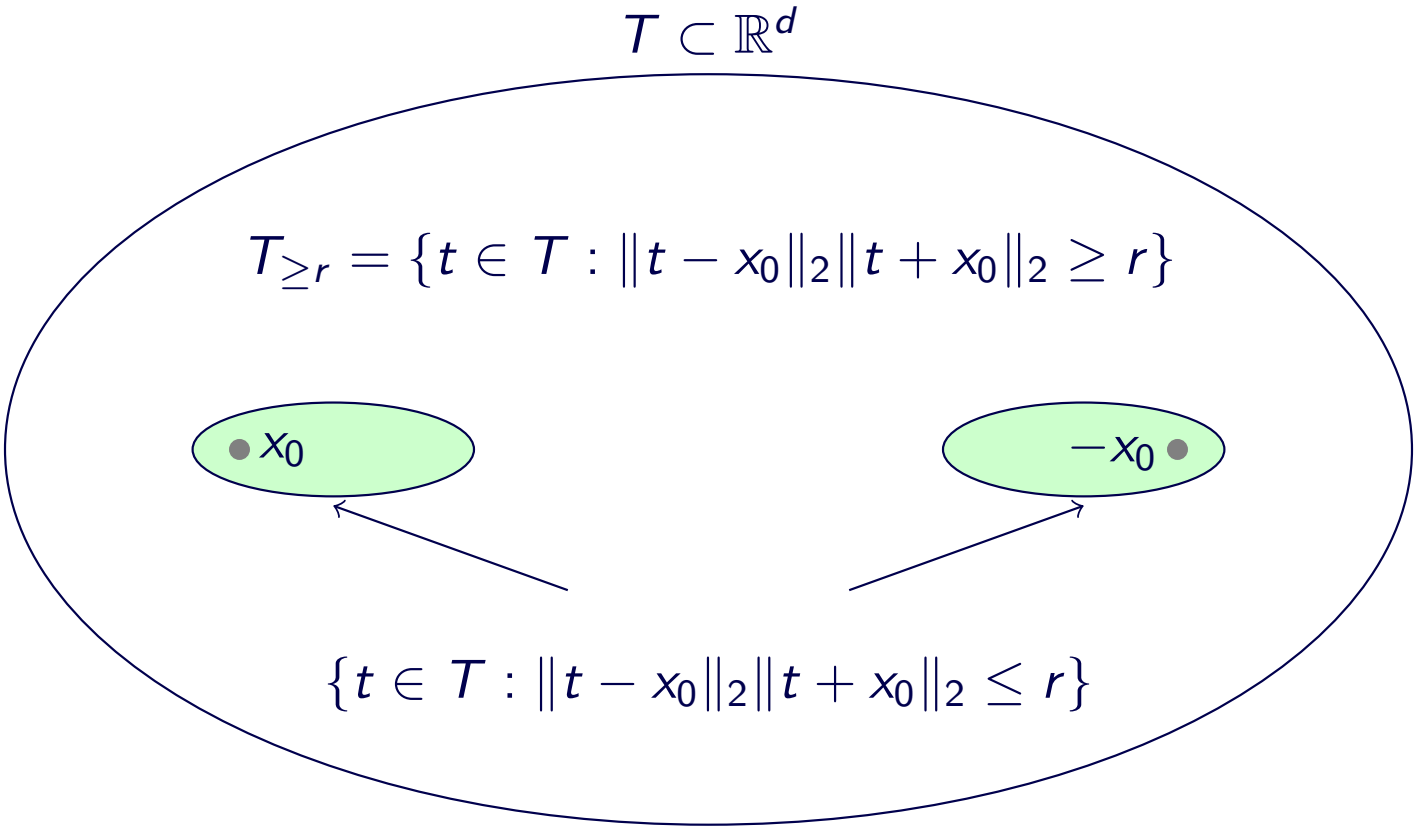
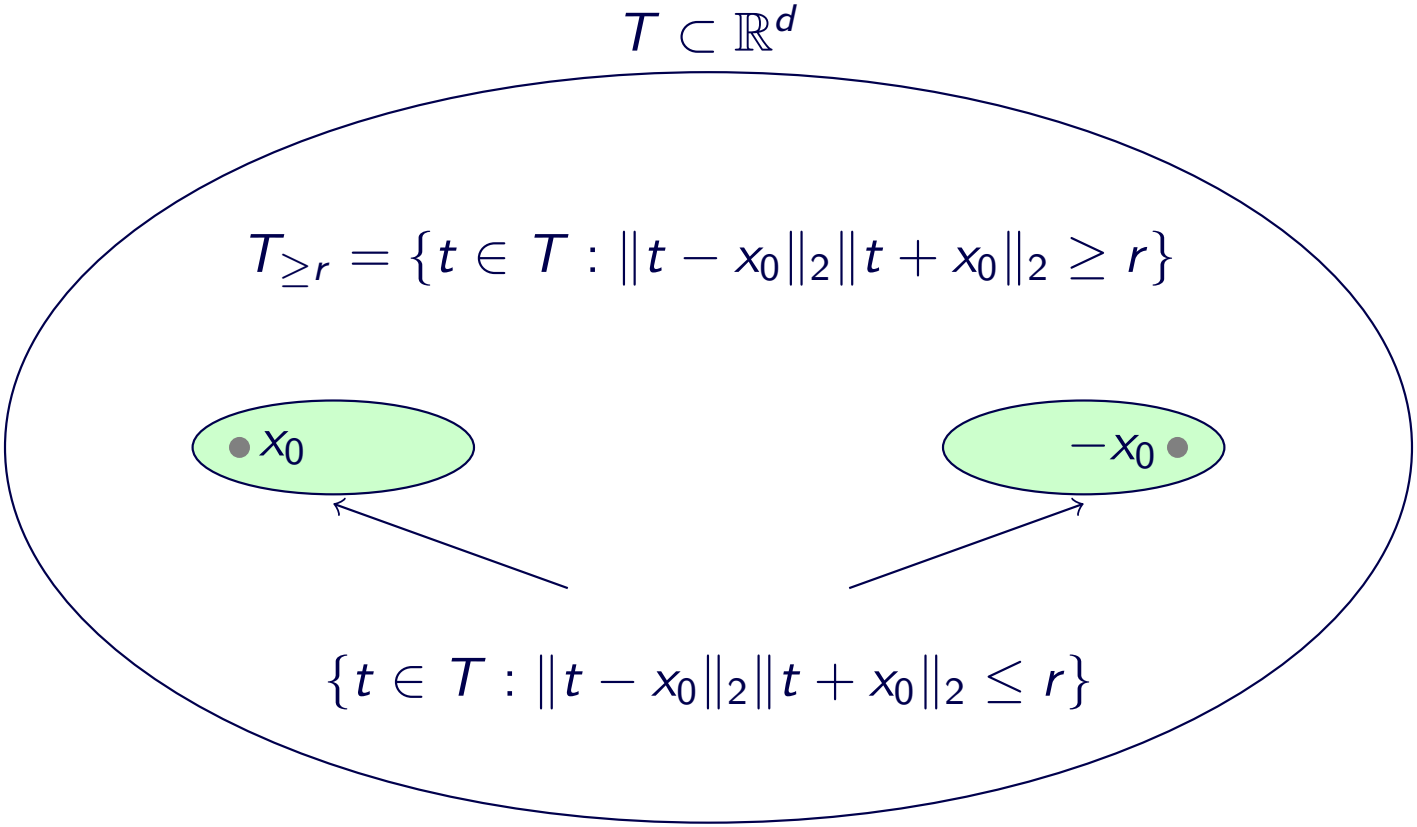


$$T_{\pm,r} = \left\{ \frac{t \pm x_0}{\|t \pm x_0\|_2} : t \in T, \|t - x_0\|_2 \|t + x_0\|_2 \geq r \right\}.$$



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Measure of complexity :

$$E_r = \max\left(\ell(T_{-,r}), \ell(T_{+,r})\right)$$

The "quadratic" + "multiplier" decomposition

Decomposition of the empirical excess loss :

$$P_N \mathcal{L}_x = \frac{1}{N} \sum_{i=1}^{N} \langle a_i, x - x_0 \rangle^2 \langle a_i, x + x_0 \rangle^2 - \frac{2\sigma}{N} \sum_{i=1}^{N} g_i \langle a_i, x - x_0 \rangle \langle a_i, x + x_0 \rangle$$

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power 3 of a subgaussian variables ($\sim \psi_{2/3}$) - control via contraction principle : a $\sqrt{\log N}$ extra term.

Control of the "quadratic" term via empirical small ball estimate

Proposition

If $\sqrt{N} \gtrsim E_r$ then w.h.p. for any $t \in T$ such that $\|t - x_0\|_2 \|t + x_0\|_2 \geq r$, there exists $I_t \subset \{1, \dots, N\}$ such that $|I_t| \gtrsim N$

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$$|\langle t - x_0, a_i \rangle \langle t + x_0, a_i \rangle| \gtrsim \|t - x_0\|_2 \|t + x_0\|_2.$$

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Sharp control of the "quadratic term" as long as $\sqrt{N} \gtrsim E_r$: "fixed point equation" for r when the noise is small.

Control of the "multiplier" process

w.h.p. for any $t \in T$ such that $\|t - x_0\|_2 \|t + x_0\|_2 \geq r$,

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then for all $t \in T_{\geq r}$, $P_N \mathcal{L}_t > 0$ but $P_N \mathcal{L}_{\hat{x}} \leq 0$ therefore, $\hat{x} \notin T_{\geq r}$:

$$\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2 \leq r.$$

Two fixed points

1. Complexity coming from the noise measured via the "multiplier process's fixed point" :

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Nevertheless, it is easier to understand a result like

$$\min\left(\|\hat{x} - x_0\|_2, \|\hat{x} + x_0\|_2\right) \leq \textit{rate}.$$

Relation between $\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2$ and $\min\left(\|\hat{x} - x_0\|_2, \|\hat{x} + x_0\|_2\right)$

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$\Rightarrow$ 2 regimes for (the localization and thus) the complexity term E_r depending on $\|x_0\|_2$.

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where

rate	$\sigma\sqrt{\log N} \leq \|x_0\|_2 r_N^*$	$\sigma\sqrt{\log N} \geq \|x_0\|_2 r_N^*$
$\|x_0\|_2 \geq v_N^*$	r_N^*	s_N^*
$\|x_0\|_2 \leq v_N^*$	r_N^*	v_N^*

Sparse vectors

$$T = W_s = \{x \in \mathbb{R}^d : |\mathrm{supp}(x)| \leq s\}$$

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$$\ell(W_s \cap rB_2^n) \sim r\sqrt{s \log\left(ed/s\right)}.$$

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Sudakov inequality is sharp : $r \log^{1/2} N\left(W_s \cap 2rB_2^d, rB_2^d\right) \sim \ell(W_s \cap rB_2^n)$

$$\Downarrow$$

ERM is minimax in linear regression and phase recovery (up to $\sqrt{\log N}$) in the noisy setup.

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Rates of convergence over W_s in linear regression and phase recovery

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Rates of convergence over W_s in linear regression and phase recovery

When $N \gtrsim s \log\left(ed/s\right)$. In linear regression :

1. if $\sigma = 0$ then w.h.p. $\hat{x} = x_0$.
2. if $\sigma > 0$, then w.h.p.

$$\|\hat{x} - x_0\|_2 \lesssim s_N^*(\sigma^{-1}) \sim \sigma\sqrt{\frac{s \log\left(ed/s\right)}{N}}.$$

In phase recovery :

1. if $\sigma = 0$ then w.h.p. $\hat{x} = x_0$ or $\hat{x} = -x_0$.
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The unit B_1^d-ball

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1. Gaussian complexity of localized sets :

$$\ell(B_1^d \cap rB_2^d) \sim \begin{cases} \sqrt{\log\left(edr^2\right)} & \text{if } d^2 r \geq 1 \\ r\sqrt{d} & \text{otherwise} \end{cases}$$

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$$\Downarrow$$

ERM is minimax in linear regression and phase recovery (up to $\sqrt{\log N}$) in the noisy case (and also in the noise free case).

Computing the three fixed points

$$r_N^*(Q) \begin{cases} \sim \left(\frac{1}{Q^2 N} \log\left(\frac{n}{Q^2 N}\right)\right)^{1/2} & \text{if } n \geq C_0 Q^2 N \\ \lesssim \frac{1}{N} & \text{if } C_1 Q^2 N \leq n \leq C_0 Q^2 N \\ = 0 & \text{if } n \leq C_1 Q^2 N. \end{cases}$$

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$$v_N^*(\zeta) \sim \begin{cases} \left(\frac{1}{\zeta^2 N} \log\left(\frac{n^3}{\zeta^2 N}\right)\right)^{1/6} & \text{if } n \geq \zeta^{2/3} N^{1/3} \\ \left(\frac{n}{\zeta^2 N}\right)^{1/4} & \text{if } n \leq \zeta^{2/3} N^{1/3}. \end{cases}$$

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Thanks for your attention