

Periodic Patterns Frequency Hopping Waveforms : from conventional Matched Filtering to a new Compressed Sensing Approach

Philippe Mesnard*, Cyrille Enderli**, Guillaume Lecué**

*Thales Systèmes Aéroportés
Elancourt, FRANCE
email: philippe.mesnard@fr.thalesgroup.com

**Thales Systèmes Aéroportés
Elancourt, FRANCE
email: cyrille-jean.enderli@ifr.thalesgroup.com

***ENSAE ParisTech
Palaiseau, FRANCE
email: guillaume.lecue@ensae.fr

Abstract: *The process conventionally used to estimate the parameters of the targets detected by the radar is matched filtering, it is optimal for the detection of a single point target. Compressed sensing would define a new treatment for the radar, also performing in the multi-target case. In addition, we seek to apply this treatment to step-frequency waveforms. We hope to get better performances, particularly in terms of tracking and recognition. We formulated the problem of compressed sensing to the radar, and generated potentially suitable waveforms. Our main interest is a very particular structure composed by periodic patterns of pulses affected to different carrier frequencies. That shape ables to get efficient reconstruction with fine resolution and low number of measurements, making more interesting than a standard step frequency waveform without periodic structure. ...*

1. Introduction

We consider as a signal a train of N pulses of duration T , transmitted with a repetition period T_R , whose carrier frequency varies from one pulse to the next. We note, for $n \in \llbracket 1; N \rrbracket$, f_n the frequency jump such as the carrier frequency of the n th emitted pulse is $f_0 + f_n$. In this type of waveforms, and in the context of conventional reconstruction processing, the frequency bandwidth is no longer defined by the inverse of the duration of a pulse, but by the maximum difference in absolute value between the carrier frequencies of two distinct pulses of the train. But the distance resolution of the radar is inversely proportional to the bandwidth of the transmitted signal [1]; we can therefore control the accuracy of the distance reconstruction by adjusting the frequency values assigned to the pulses. This is advantageous for target identification missions, which require fine range resolution. Generally, the shape of these signals leaves a large margin

of choice in their definition parameters and make them adaptable to different types of missions. Moreover, these waveforms are relatively simple to generate and a radar emitting such a signal is more resistant to jamming and listening. It is in this perspective that we focus our study towards this type of waveforms.

The different frequencies assigned to the signal pulses are defined from a frequency quantum Δ_f and chosen from a set of N frequencies $\{k\Delta_f; k = 1\dots N\}$. Instead of using each of these frequencies, we seek to avoid forbidden frequencies, in particular scrambled frequencies, and to intercept forbidden frequency bands due to compatibility with other spectrum users (e.g reserved communication channels). In order to avoid this while maintaining such a fine range resolution, we can choose to use only a limited number $F < N$ of distinct frequencies in the set $\{k\Delta_f; k = 1\dots N\}$ from which we retain the lowest and the highest. Since we always want to emit N pulses to satisfy the energy balance requirements, here is how we proceed to generate this new waveform model:

- the radar emits a first pattern of F pulses, of period T_R and of duration T . The frequency of the j th pulse, $j \in \llbracket 1; F \rrbracket$ is $f_0 + f_j$, where $f_j = p(j)\Delta_f$. p denotes here an injective application of $\llbracket 1; F \rrbracket$ in $\llbracket 1; N \rrbracket$, such that there exist two elements j_1 and j_2 of $\llbracket 1; F \rrbracket$ satisfying $p(j_1) = 1$ and $p(j_2) = N$.
- This pattern is repeated periodically until reaching the total number of expected pulses. If U is the total number of patterns of F pulses, then $UF = N$.

We write this signal as follow :

$$s(t) = e^{i2\pi f_0 t} \sum_{u=0}^{U-1} \sum_{j=0}^{F-1} e^{i2\pi f_{j+1}(t-(uF+n)T_R)} \Pi_T(t - nT_R) \quad (1)$$

In this expression, the summation variable u designates a pattern index while j denotes a frequency index. An example of this type of signal can be represented by the figure below, showing in the form of a time-frequency graph the (truncated) succession of periodic patterns composed of three pulses assigned to two by two distinct frequencies.

We will now seek to implement the treatment presented in the previous section to this kind of waveforms. The main principles of classical matched filtering [2] process will be recalled, and its limits on a periodic patterns hopping waveform will be showed. Finally, we introduce the concept of Compressed Sensing (CS) [3] and explain the advantages we expect from its application to electromagnetic signals previously introduced.

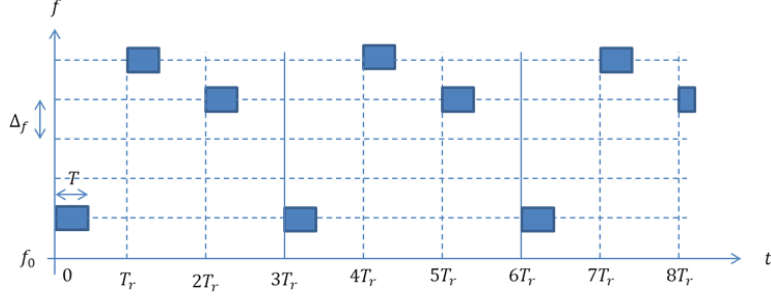


Figure 1: Time-frequency graph of a periodic frequency hopping waveform

2. Implementation of matched filtering on studied waveforms

2.1. Processing set up

Consider a single point target of parameters (τ, ν) and amplitude α , which reflects the waveform $s(t)$ emitted by the radar. The expression of the received signal is deduced by the relation $r(t) = \alpha s\left(t - \left(\tau - \frac{\nu}{f_0}t\right)\right) + w(t)$ [2].

We will take into account three hypotheses [2] [1] to apply this formula to our waveform : the classical narrow-band hypothesis ; the fact that Doppler migrations are negligible, implying $e^{i2\pi\frac{2\nu_r}{c}f_{j+1}t} \approx 1$ and the fact that Doppler modulation can be considered constant along a pulse, implying $e^{i2\pi\nu t} = e^{i2\pi\nu(uF+j)T_R}$.

Finally, assuming that these three hypotheses are satisfied, we can write the received signal as follow : $r(t) = \alpha s(t - \tau)e^{i2\pi\nu t} + w(t)$. The parameters range delay and Doppler of the targets that generated the signal received by the radar are reconstructed in a bi-dimensional discrete grid. The testable delays, $\{\tau_l = \frac{l}{B}, 0 \leq l \leq BT\}$, figure on the Y axis, whereas the testable Doppler frequencies, $\{\nu_m = \frac{m}{NT_R}, 0 \leq m \leq N\}$, lie on the X axis. Calculation leads to a scheme illustrating matched filtering applied to the studied waveform, using the Euclidean division of m by U : $m = qU + r, r \in \llbracket 0; U - 1 \rrbracket$ and a discrete Fourier transform with U elements, except that $m \in \llbracket 0; N - 1 \rrbracket$ and $N = U * F$: $FFT_F\{\Gamma(u)\}_{\frac{m}{U}}$ with $\Gamma_j(u) = e^{-i2\pi\frac{m[U]}{N}j} \int_{\frac{l}{B}+(uF+j)T_R}^{\frac{l}{B}+(uF+j)T_R+T} r(t)e^{-i2\pi f_{j+1}(t-\frac{l}{B})} dt$ and $FFT_U\{\Omega\}_{m[U]}$ with $\Omega_u = FFT_F\{\Gamma(u)\}_{\lfloor \frac{m}{U} \rfloor}$.

We can convolute again to calculate the different integrals, so that the treatment can be performed more efficiently. We can recap the previous equations defining matched filtering applied to periodic patterns waveforms by the underlying scheme :

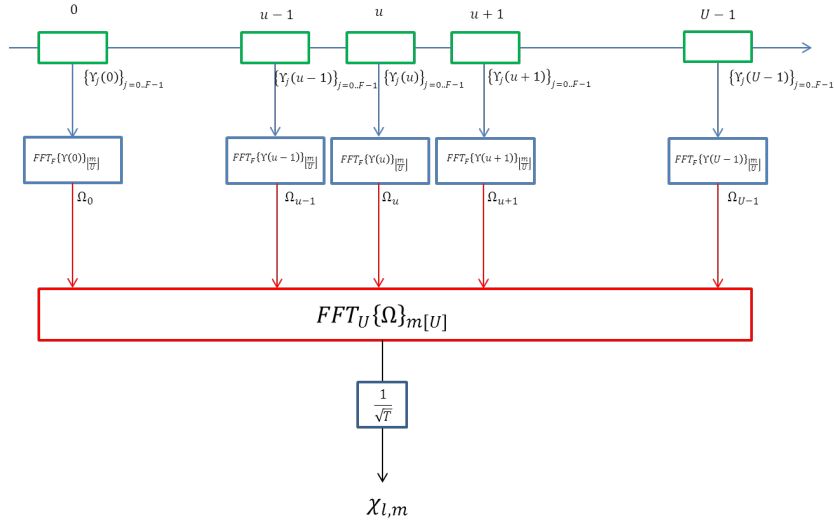


Figure 2: Matched filtering applied to a waveform composed by U periodic patters of F electromagnetic pulses

2.2. Numerical simulations of matched filtering on studied waveforms

We can see in next Figure the result of the adapted filtering we obtained with *Matlab*, from the detection of a single point target at delay index $l = 1225$ and Doppler index $m = 115$, neglecting the Doppler migrations and without adding any noise. The periodic frequency hopping signal chosen to run the simulations is composed of 50 patterns of 4 pulses affected to different frequencies.

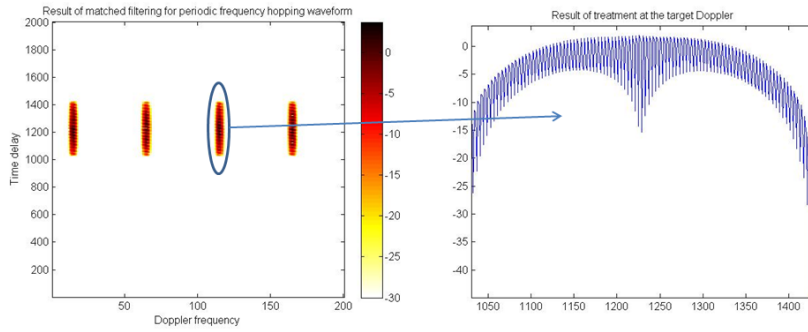


Figure 3: Matched filtering applied to an incomplete and periodic waveforms

We observe the repetition of a reconstruction pattern four times over the range of delays intercepting a target, probably due to the gaps in the spectrum which is not used continuously. Besides, the cut of the response for the frequency of the target, visible on the right part of the figure above, shows numerous local maxima and relatively close to the absolute maximum. The distinction of this one from the other maxima will be all the less guaranteed when we take into account the thermal noise of the radar, and when several targets on the same range of delays reflect the radar signal, since each reflecting contribution is then added in the reconstruction grid.

This waveform with very promising industrial advantages for the radar appears then very little adapted to the conventional treatment.

2.3. Assessment of the limits of conventional treatment

It turns out that the matched filtering we studied here, and its use on the waveforms we currently use have the following limitations:

- The matched filtering is optimal only in the case where we detect at most one point target. For example, we observe a deterioration in detection performance in the presence of multiple close targets. We can mention in particular the desensitization of treatments CFAR (Constant False Alarm Rate)
- We currently do not have a processing and a suitable waveform that allow us optimal use of radar time. For example in Non-Cooperative Target Recognition (NCTR) modes, we would like to have an improvement margin for tracking and recognition. It is in this perspective in particular that we are directing our work towards the use of step frequency waveforms previously described. However, the application of matched filtering on these waveforms is not satisfactory for the purpose we are seeking.

In view of these limitations, the strengths we are seeking in the implementation of a new radar treatment on frequency hopping waves are as follows:

- Be able to detect several targets, generally elongated and no longer punctual.
- Have a versatile waveform, optimized compared to a certain alternative treatment that we present in the following part, which can provide the joint functions of tracking and recognition, and possibly standby function.
- Be able to obtain an on-the-fly estimation of the parameters of the detected targets, and thus optimize the use of radar time.

The alternative treatment that is at the heart of our study is based on a theoretical framework different from that of adapted filtering. We are going to explain what this one is.

3. Compressed Sensing processing set up for airborne radar

3.1. Principle and interpretation

The concept of CS was defined by E.Candès and D.Donoho in 2004 [3] [4]. This theory aims at the reconstruction of particular signals, called sparse. In practice, such a signal is represented by

a large vector of size P , which we will note α , the majority of the coefficients are zero. We note s the number of non-zero elements of α , such as $s \ll P$, corresponding to the pseudo-norm L_0 of vector α . We write $\|\alpha\|_0 = s$.

The theory shows that such signals can be sampled linearly with a very low sampling rate. We are therefore looking for a basis in which we can define the information system that we are looking for and satisfying the assumptions of CS. To do this, let us take the grid used for targets reconstruction : if we concatenate the columns of this table one above the others, we obtain a vector-column, constituted of N blocks of BT elements. Each block corresponding to a frequency tested; within a block, an index represents a tested time.

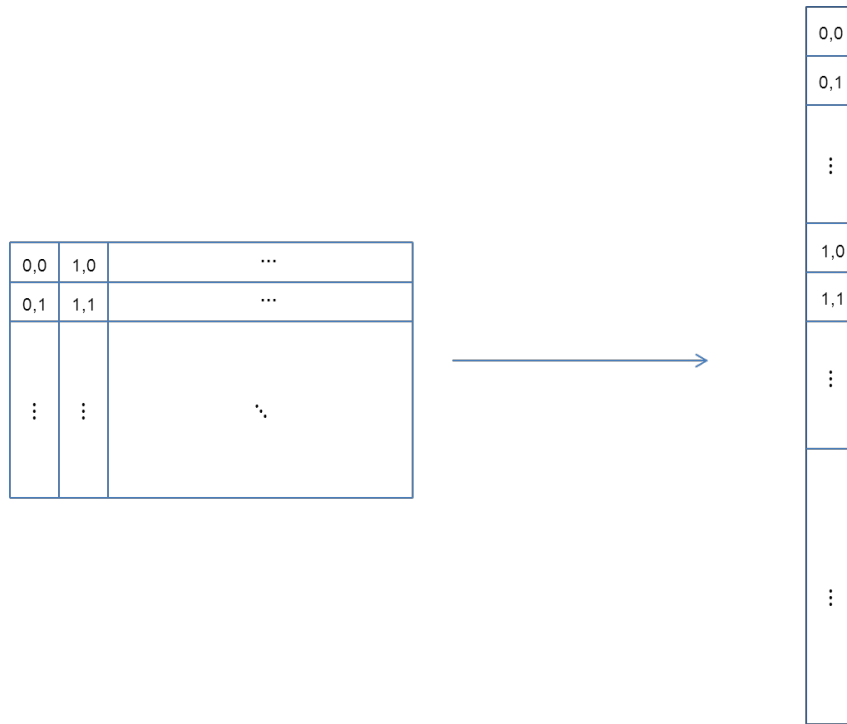


Figure 4: Reconstruction grid recast into a vector for CS processing

Let $k \in \llbracket 0; N * BT - 1 \rrbracket$ be the index of one of the coefficients of the vector, and consider its Euclidean division by N : $k = mN + l$. $m \in \llbracket 0; N - 1 \rrbracket$ corresponds to the index of the block in which is located the k -th element of the vector. The Doppler frequency associated with this element is therefore $\nu_m = \frac{m}{NT_R}$. $l \in \llbracket 0; BT - 1 \rrbracket$ is the number of the element k within the block of subscript m , it is associated with the delay $\tau_l = \frac{l}{B}$.

We can therefore find both the delay and the frequency associated with a coefficient of the vector directly from its coordinates. The values of these coefficients depend on the amplitude of the corresponding target, it is zero in the absence of a target characterized by these parameters. s is here the total number of points detected by the radar, and α constitutes an inventory of the present targets, with their parameters and their amplitude. In this sense, the vector α can be interpreted as the "scenario" of the scene observed by the radar.

We generally have a small number of targets to detect in relation to the search space thus defined, the signals received are therefore sparse and the CS thus proves to be promising. It should also be noted that this formulation takes into account the possible presence of several targets, unlike matched filtering. It remains to be seen how the studied theory proposes to reconstruct such a vector.

CS postulates the existence of a system of measurements, which we represent by a vector denoted u , of dimension M , which can be expressed linearly from the information sought and an additional Gaussian noise w from the radar : $u = \Phi\alpha + w$.

The major assumption in this writing is that the measurements are significantly less numerous than the number of elements of α , which translates to $M \ll NBT$ in our case. Φ , the measurement matrix of the radar, is thus rectangular, of dimension $M \times NBT$. A line of Φ is associated with a measurement, while a column is associated with a pair of time-frequency parameters tested. We will show later that the expression of this matrix depends on the characteristics of the waveform used; it is therefore known by the user of the radar. The purpose of the reconstruction is to find the vector α containing all the information of interest to us, knowing that we have the measured value u and that we know the measurement matrix Φ . Such a vector is solution of the matrix equation written above.

But Φ has more columns than rows, and this equation has an infinity of solutions. CS states that, under certain conditions verified by the measurement matrix, the solution sought is that which has the least non-zero elements. This amounts to determining the solution of the problem of minimization under constraints below, in case we do not take the noise into account :

$$\min_{\alpha \in \mathbb{C}^{NBT_R}} \|\alpha\|_0 \quad \text{s.t.} \quad u = \Phi\alpha.$$

The constraints mean that we are looking for our solution among the vectors satisfying the measurement equation, and the minimization of the criterion implies that we are looking for the most sparse signal.

The disadvantage of the above problem is that it is non-convex, and the duration of its numerical resolution grows exponentially with the size of the data. However, if the product of the measurement matrix and parsimony base checks some property, Restricted Isometry Property (RIP), then we can replace the pseudo-norm above with the standard norm l_1 , and solve a convex program named Basis Pursuit (BP), it can be solved by efficient iterative algorithms. In presence of noise, the program to be solved for the radar writes :

$$\min_{\alpha \in \mathbb{C}^{NBT_R}} \|\alpha\|_1 \quad \text{s.t.} \quad \|u - \Phi\alpha\|_2^2 \leq \delta$$

This problem is called Basis Pursuit DeNoise (BPDN). δ is a threshold that depends on the variance of the noise.

The next objective sought to be able to apply the principle of CS to the radar is therefore to find a measurement system u of size smaller than the size of the reconstruction grid, making it

possible to rewrite a linear matrix equation with the desired signal and a matrix of measurements depending on the emitted waveform.

3.2. Measures and reconstruction

We seek here to implement, using the techniques described above, a new treatment on the studied step frequency waveforms, in order to find the targets detected by the radar. The concept of CS allows us to assume the existence of several targets detectable simultaneously by the radar. The writing of the CS problem is based on the definition of exhaustive statistics for the target amplitude parameters, by calculating the probability density of the received signal with respect to the measurement of the noise.

The CS problem has for parameters the vector of measurements u , of size M , and the measurement matrix Φ , of dimensions $M \times NBT$. The associated search space is $\left\{ \left(\frac{m}{NT}, \frac{l}{B} \right) : m \in \llbracket 0; N-1 \rrbracket, l \in \llbracket 0; BT-1 \rrbracket \right\}$. The solution is a vector α , of size NBT . To numerically obtain the associated resolution grid, we operate in inverse fashion to the previous sub-part, concatenating the N blocks of α next to one another. We can explain the coefficients of the measurement matrix. First, we assume that all the hypotheses formulated allow us to simplify the expression of the received signal are satisfied.

We denote $\Phi = \{\Phi_{m,l}(j)\}_{j \in \llbracket 0; M-1 \rrbracket, m \in \llbracket 0; N-1 \rrbracket, l \in \llbracket 0; BT-1 \rrbracket}$. On the one hand, m is a block index of BT columns, associated with a frequency of the reconstruction grid and l is the index of the column tested within the block m , associated with a delay of the reconstruction grid. On the other hand, j is a line index associated with a measurement and, equivalently, a pulse of the radar due to the writing of the exhaustive statistics. Special advantages of the waveform used here appear now. First, a smaller frequency bandwidth in this signal can lead to the same range resolution as before. And second, only the first F data obtained from exhaustive statistics are necessary to get a CS problem for the radar with a measurement matrix satisfying suitable algebraic properties.

Thus, although we transmit $N = U * F$ pulses in total, we only perform F measurements, each of them being assigned to one of the F pulses of a pattern of the waveform or, equivalently, to one of the F distinct frequencies used. As a result, the matrix of measurements Φ always has NBT columns but only $M = F$ lines. The expression of the coefficients of this matrix is then given by injection p of $\llbracket 1; F \rrbracket$ in $\llbracket 1; N \rrbracket$ defining the sequence of carrier frequencies assigned to the pulses. This will affect the search for a waveform of this type, suitable for processing using CS.

$$\Phi_{m,l}(j) = e^{i\frac{2\pi}{N}(jm-p(j+1)l)} \quad (2)$$

It should be noted that the choice of this waveform would make it possible to perform fewer measurements than for the frequency evasion waveform, provided that it is compatible with all

the hypotheses allowing us to apply the CS principle. In fact F cannot be taken too small, otherwise no reconstruction is possible at all. Diagram for rectangular measurement matrices with F lines defined from random injections p give the relation between the number s of reconstructible point targets and the necessary number F of measures to be stored.

The figure below obtained from *Matlab* numeric simulations shows a successful reconstruction of five point targets by using a CS program called *spgl1*. It has been obtained from a step frequency waveform composed of two identical and successive train of 100 pulses. This yields to a reconstruction with fine resolution and required energy balance associated to the total number of 200 pulses in the signal and to a lower storage of 100 measurements, instead of 200.

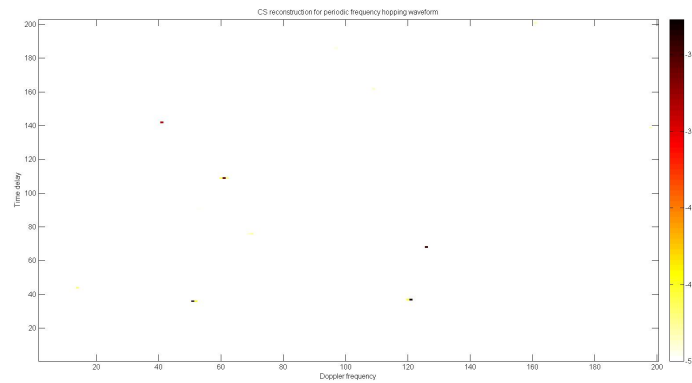


Figure 5: Reconstruction of target points by CS

4. Conclusion

Classical pulse train of identical carrier frequency conventionally emitted by airborne radar to detect and track targets are not adapted to identification missions, that require fine range and Doppler resolutions. The general family of step frequency waveforms offers better perspectives since the agility of their definition parameters permits to control the shape and structure of the grid used for reconstruction of targets. Due to further operational considerations, periodic patterns frequency hopping waveforms are of special interest for airborne detection, location and identification under strong electromagnetic requirements. However, conventional matched filtering applied to such waveforms yields to intractable reconstruction results. To solve that limitation, it is necessary to find another treatment, adapted to the special structure of the radar signal. Compressed Sensing principle has been retained here and gives a suitable new reconstruction processing. Moreover, it ables to take into account multi-targets scenes better than matched filtering. And finally, the special shape of periodic patterns permits lower data storage at constant resolution and energy balance. Nevertheless, the minimal number of measurements in CS is relatively high if we expect to reconstruct exactly several targets. This point can limit the effectiveness of the process since we can't emit too many pulses, and this explains the necessity of further algebraic tools such as phase transition diagrams to generate suitable signals.

References

- [1] F. Le Chevalier, "Principles of Radar and Sonar Signal Processing", Masson, Paris, 1989
- [2] N. Levanon and E. Mozeson, "Radar Signals", *Wiley Interscience, Hoboken Energy*, 2004
- [3] S. Foucart and H. Rauhut, "A Mathematical Introduction to Compressive Sensing", 2013
- [4] E. J. Candès and M. B. Wakin, "An Introduction to Compressive Sampling", *IEEE Signal Processing Magazine*, vol.21, pp.21-30,, March 2008