Learning with semi-definite programming: statistical bounds based on fixed point analysis and excess risk curvature

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Abstract

Many statistical learning problems have recently been shown to be amenable to Semi-Definite Programming (SDP), with *community detection* and *clustering in Gaussian mixture models* as the most striking instances Javanmard et al. (2016). Given the growing range of applications of SDP-based techniques to machine learning problems, and the rapid progress in the design of efficient algorithms for solving SDPs, an intriguing question is to understand how the recent advances from empirical process theory and Statistical Learning Theory can be leveraged for providing a precise statistical analysis of SDP estimators.

In the present paper, we borrow cutting edge techniques and concepts from the Learning Theory literature, such as fixed point equations and excess risk curvature arguments, which yield general estimation and prediction results for a wide class of SDP estimators. From this perspective, we revisit some classical results in community detection from Guédon and Vershynin (2016) and Fei and Chen (2019), and we obtain statistical guarantees for SDP estimators used in signed clustering, group synchronization (for both multiplicative and additive models) and MAX-CUT. Our theoretical findings are complemented by numerical experiments for each of the three problems considered, showcasing the competitiveness of the SDP estimators.

Keywords: Semi-Definite Programming, Statistical Learning, Group Synchronization, Signed Clustering.

Contents

1. Introduction

Many statistical learning problems have recently been shown to be amenable to Semi-Definite Programming (SDP), with *community detection* and *clustering in Gaussian mixture models* as the most striking instances where SDP performs significantly better than other current approaches Javanmard et al. (2016). SDP is a class of convex optimization problems generalising linear programming to linear problems over semi-definite matrices Todd (2001), Wolkowicz et al. (2012), Boyd and Vandenberghe (2004), and which was proved to be an important tool in the computational approach to difficult challenges in automatic control, combinatorial optimization, polynomial optimization, data mining, high-dimensional statistics and the numerical solution to partial differential equations. The goal of the present paper is to introduce a new fixed point approach to the statistical analysis of SDP-based estimators and illustrate our method on four current problems of interest, namely MAX-CUT, *community detection*, *signed clustering*, *angular group synchronization*. Our aim is to show that all those estimators can be viewed as special instances of ERM, and can then benefit from the very large litterature on that subject. The rest of this section gives some historical background and presents the mathematical definition of SDP-based estimators.

1.1 Historical background

SDP is a class of optimization problems which includes linear programming as a particular case and can be written as the set of problems over symmetric (resp. Hermitian) positive semi-definite matrix variables, with linear cost function and affine constraints, i.e. optimization problems of the form

$$\max_{Z \succeq 0} \left(\langle A, Z \rangle : \langle B_j, Z \rangle = b_j \text{ for } j = 1, \ldots, m \right), \tag{1}$$

where $A, B_1, \ldots, B_m$ are given matrices. SDPs are convex programming problems which can be solved in polynomial time when the constraint set is compact and it plays a paramount role in a large number of convex and non-convex problems, for which they often appear as a convex relaxation Anjos and Lasserre (2011). We will sometimes use the notation $\mathbb{S}_{n,+}$ (resp. $\mathbb{S}_{n,-}$) for the cone of positive semi-definite matrices (resp. negative semi-definite matrices).

1.1.1 EARLY HISTORY

Early use of Semi-Definite programming to statistics can be traced back to Scobey and Kabe (1978) and Fletcher (1981). In the same year, Shapiro used SDP in factor analysis Shapiro (1982). The study of the mathematical properties of SDP then gained momentum with the introduction of Linear Matrix Inequalities (LMI) and their numerous applications in control theory, system identification and signal processing. The book Boyd et al. (1994) is the standard reference of these type of results, mostly obtained in the 90's.

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1.1.2 The Goemans-Williamson SDP relaxation of Max-Cut and its legacy

A notable turning point is the publication of Goemans and Williamson (1995) where SDP was shown to provide a 0.87 approximation to the NP-Hard problem known as Max-Cut. The Max-Cut problem is a clustering problem on graphs which consists in finding two complementary subsets S and S^c of nodes such that the sum of the weights of the edges between S and S^c is maximal. In Goemans and Williamson (1995), the authors approach this difficult combinatorial problem by using what is now known as the Goemans-Williamson *SDP relaxation* and use the Choleski factorization of the optimal solution to this SDP in order to produce a randomized scheme achieving the .87 bound in expectation. Moreover, this problem can be seen as a first instance where the Laplacian of a graph is employed in order to provide an optimal bi-clustering in a graph and certainly represents the first chapter of a long and fruitful relationship between clustering, embedding and Laplacians. Other SDP schemes for approximating hard combinatorial problems are, to name a few, for the graph coloring problem Karger et al. (1998), for satisfiability problem Goemans and Williamson (1995, 1994). These results were later surveyed in Lemaréchal et al. (1995); Goemans (1997) and Wolkowicz (1999). The randomized scheme introduced by Goemans and Williamson was then further improved in order to study more general Quadratically Constrained Quadratic Programmes (QCQP) in various references, most notably Nesterov (1997); Zhang (2000) and further extended in He et al. (2008). Many applications to signal processing are discussed in Olsson et al. (2007), Ma (2010); one specific reduced complexity implementation in the form of an eigenvalue minimization problem and its application to binary least-squares recovery and denoising is presented in Chrétien and Corset (2009).

1.1.3 Relaxation of machine learning and high-dimensional statistical estimation problems

Applications of SDP to problems related with machine learning is more recent and probably started with the SDP relaxation of K-means in Peng and Xia (2005); Peng and Wei (2007) and later in Ames (2014). This approach was then further improved using a refined statistical analysis by Royer (2017) and Giraud and Verzelen (2018). Similar methods have also been applied to community detection Hajek et al. (2016); Abbe et al. (2015) and for the weak recovery viewpoint, Guédon and Vershynin (2016). This last approach was also re-used via the kernel trick for the point cloud clustering Chrétien et al. (to appear). Another incarnation of SDP in machine learning is the extensive use of nuclear norm-penalized least-square costs as a surrogate for rank-penalization in low-rank recovery problems such as matrix completion in recommender systems, matrix compressed sensing, natural language processing and quantum state tomography; these topics are surveyed in Davenport and Romberg (2016).

The problem of manifold learning was also addressed using SDP and is often mentioned as one of the most accurate approaches to the problem, let aside its computational complexity; see Weinberger et al. (2005); Weinberger and Saul (2006b,a); Hegde et al. (2012). Connections with the design of fast converging Markov-Chains were also exhibited in Sun et al. (2006).

In a different direction, A. Singer and collaborators have recently promoted the use of SDP relaxation for estimation under group invariance, an active area with many applica-

tions Singer (2011); Bandeira et al. (2014). SDP-based relaxations have also been considered in Cucuringu (2015) in the context of synchronization over $\mathbb{Z}_2$ in signed multiplex networks with constraints, and Cucuringu (2016) in the setting of ranking from inconsistent and incomplete pairwise comparisons where an SDP-based relaxation of angular synchronization over SO(2) outperformed a suite of state-of-the-art algorithms from the literature. Phase recovery using SDP was studied in e.g. Waldspurger et al. (2015) and Demanet and Hand (2014). An extension to multi-partite clustering based on SDP was then proposed in Karger et al. (1998). Other important applications of SDP are, information theory Lovász (1979), estimation in power networks Lavaei and Low (2011), quantum tomography Mazziotti (2011), Gross et al. (2010) and polynomial optimization via Sums-of-squares relaxations Lasserre (2015); Blekherman et al. (2012). Sums of squares relaxations were recently applied to statistical problems in De Castro et al. (2019); Hopkins (2018); de Castro et al. (2017). Extension to the field of complex numbers, with $\langle \cdot, \cdot \rangle$ denoting the Hermitian inner product, has been less extensively studied but has many interesting applications and comes with efficient algorithms Goemans and Williamson (1995); Gilbert and Josz (2017).

1.2 Mathematical formulation of the problem

The general problem we study can be stated as follows. Let A be a random matrix in $\mathbb{R}^{n \times n}$ and $\mathcal{C} \subset \mathbb{R}^{n \times n}$ be a constraint. The object that we want to recover, for instance, the community membership vector in community detection, is related to an *oracle* defined as

$$Z^* \in \underset{Z \in \mathcal{C}}{\operatorname{argmax}} \, \langle \mathbb{E}A, Z \rangle, \tag{2}$$

where $\langle A, B \rangle = \operatorname{Tr}(A\bar{B}^\top) = \sum A_{ij}\bar{B}_{ij}$ when $A, B \in \mathbb{C}^{n \times n}$ where $\bar{z}$ is the conjugate of $z \in \mathbb{C}$. We would like to estimate Z^*, from which we can ultimately retrieve the object that really matters to us (for instance, by considering a singular vector associated to the largest singular value of Z^*). To that end, we consider the following natural estimator of Z^* given by:

$$\hat{Z} \in \underset{Z \in \mathcal{C}}{\operatorname{argmax}} \, \langle A, Z \rangle, \tag{3}$$

which is simply obtained by replacing the unobserved quantity $\mathbb{E}A$ by the observation A.

As pointed out, in many situations, Z^* is not the object we want to estimate, but there is a straightforward relation between Z^* and this object. For instance, consider the community detection problem, where the goal is to recover the class community vector $x^* \in \{-1, 1\}^n$ of n nodes. Here, when $\mathcal{C}$ is well chosen, there is a close relation between Z^* and x^*, given by $Z^* = x^*(x^*)^\top$. We therefore need a final step to estimate x^* using $\hat{Z}$, for instance, by letting $\hat{x}$ denote a top eigenvector of $\hat{Z}$, and then using the Davis-Kahan "sin-Theta" Theorem Davis and Kahan (1970); Yu et al. (2015) to control the estimation of x^* by $\hat{x}$ from the one of Z^* by $\hat{Z}$.

When the constraint $\mathcal{C}$ is of the form $\mathcal{C} = \{Z \in \mathbb{R}^{n \times n} : Z \geq 0, \langle Z, B_j \rangle = b_j, j = 1, \ldots, m\}$ where $B_1, \ldots, B_m \in \mathbb{R}^{n \times n}$ and $Z \geq 0$ is notation for " Z is positive semidefinite" then (3) is a semidefinite programming (SDP) Boyd and Vandenberghe (2004).

Goal of the paper. The aim of the present work is to present a general approach to the study of the statistical properties of SDP-based estimators defined in (3). In particular,

using our framework, one is able to obtain new (non-asymptotic) rates of convergence or exact reconstruction properties for a wide class of estimators obtained as a solution of a semidefinite program like (3). Specifically, our goal is to show that the solution to (3) can be analyzed in a statistical way when $\mathbb{E}A$ is only partially and noisily observed in A. Even though the constraint $\mathcal{C}$ may not necessarily be the intersection of the set of PSD (or Hermitian) matrices with linear spaces – such as in the definition of SDP – in the following, a solution $\hat{Z}$ of (3) will be called a SDP estimator because in all our examples $\hat{Z}$ will be solution of a SDP. But for the sake of generality we will only assume only minimal requirement on the shape of $\mathcal{C}$. We also illustrate our results on a number of specific machine learning problems such as various forms of *clustering problems* and *angular group synchronization*. Three out of the four examples worked out here are concerned with real-valued matrices. Only the angular synchronization problem is approached using complex matrices.

Our goal here is to show that various problems can be analyzed in the same way. Our point of view is to look at Z^* as an oracle and $\hat{Z}$ as an ERM and therefore to analyze the problem from this perspective, and in particular we can benefit from the extensive literature on the statistical properties of ERMs. Our approach reveal a general methodology and its two crucial points which are the local curvature of the excess risk and the computation of a complexity fixed point.

2. Main general results for the statistical analysis of SDP estimators

From a statistical point of view, the task remains to estimate in the most efficient way the oracle Z^*, and to that end $\hat{Z}$ is our candidate estimator. The point of view we will use to evaluate how far $\hat{Z}$ is from Z^* is coming from the Learning Theory literature. We therefore see $\hat{Z}$ as an empirical risk minimization (ERM) procedure built on a single observation A, where the loss function is the linear one $Z \in \mathcal{C} \to \ell_Z(A) = -\langle A, Z \rangle$, and the oracle Z^* is indeed the one minimizing the risk function $Z \in \mathcal{C} \to \mathbb{E}\ell_Z(A)$ over $\mathcal{C}$. Having this setup in mind, we can use all the machinery developed in Learning Theory (see for instance Vapnik (1998); Koltchinskii (2006); Massart (2007); van de Geer (2000)) to obtain rates of convergence for the ERM (here $\hat{Z}$) toward the oracle (here Z^*).

There is one key quantity driving the rate of convergence of the ERM: a fixed point complexity parameter. This type of parameter carries all the statistical complexity of the problem, and even though it is usually easy to set up, its computation can be tedious since it requires to control, with large probability, the supremum of empirical processes indexed by "localized classes". We now define this complexity fixed point related to the problem we are considering here.

Definition 1 *Let* $0 < \Delta < 1$. *The fixed point complexity parameter at deviation* $1 - \Delta$ *is*

$$r^*(\Delta) = \inf \left(r > 0 : \mathbb{P}\left[\sup_{Z \in \mathcal{C}: \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r} \langle A - \mathbb{E}A, Z - Z^* \rangle \leqslant (1/2)r \right] \geqslant 1 - \Delta \right). \quad (4)$$

Fixed point complexity parameters have been extensively used in Learning Theory since the introduction of the localization argument Massart (2007); Koltchinskii (2011); van de Geer (2000); Birgé and Massart (1993). When they can be computed, they are preferred to the

(global) analysis developed by Chervonenkis and Vapnik Vapnik (1998) to study ERM, since the latter analysis always yields slower rates given that the Vapnik-Chervonenkis bound is a high-probability bound on the non-localized empirical process $\sup_{Z \in \mathcal{C}} \langle A - \mathbb{E}A, Z - Z^* \rangle$, which is an upper bound for $r^*(\Delta)$ since $\{Z \in \mathcal{C} : \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r\} \subset \mathcal{C}$. The gap between the two global and local analysis can be important since fast rates cannot be obtained using the VC approach, whereas the localization argument resulting in fixed points such as the one in Definition 1 may yield fast rates of convergence or even exact recovery results.

An example of a Vapnik-Chervonenkis's type of analysis of SDP estimators can be found in Guédon and Vershynin (2016) for the community detection problem. An improvement of the latter approach has been obtained in Fei and Chen (2019) thanks to a localization argument – even though it is not stated in these words (we elaborate more on the two approaches from Guédon and Vershynin (2016) and Fei and Chen (2019) in Section 3). Somehow, a fixed point such as (4) is a sharp way to measure the statistical performances of ERM estimators and in particular for the SDP estimators that we are considering here. They can even be proved to be optimal (in a minimax sense) when the noise $A - \mathbb{E}A$ is Gaussian Lecué and Mendelson (2013) and under mild conditions on the complexity of $\mathcal{C}$.

Before stating our general result, we first recall a definition of a minimal structural assumption on the constraint $\mathcal{C}$.

Definition 2 *We say that the set $\mathcal{C}$ is star-shaped in Z^* when for all $Z \in \mathcal{C}$, the segment $[Z, Z^*]$ is in $\mathcal{C}$.*

This is a pretty mild assumption satisfied, for instance, when $\mathcal{C}$ is convex, which is the setup we will always encounter in practical applications, given that SDP estimators are usually introduced after a "convex relaxation" argument. Our main general statistical bound on SDP estimators is as follows.

Theorem 1 *We assume that the constraint $\mathcal{C}$ is star-shaped in Z^*. Then, for all $0 < \Delta < 1$, with probability at least $1 - \Delta$, it holds true that $\langle \mathbb{E}A, Z^* - \hat{Z} \rangle \leqslant r^*(\Delta)$.*

Theorem 1 applies to any type of setup where an oracle Z^* is estimated by an estimator $\hat{Z}$ such as in (3). Its result shows that $\hat{Z}$ is almost a maximizer of the true objective function $Z \to \langle \mathbb{E}A, Z \rangle$ over $\mathcal{C}$ up to $r^*(\Delta)$. In particular, when $r^*(\Delta) = 0$, $\hat{Z}$ is exactly a maximizer such as Z^* and, in that case, we can work with $\hat{Z}$ as if we were working with Z^* without any loss. Then, in this "exact reconstruction case", the information contained about A on $\mathbb{E}[A]$ is enough for knowing Z^* exactly.

Theorem 1 may be applied in many different settings; in the following, we study four such instances. We will apply Theorem 1 (or one of its corollary stated below) to several popular problems in the networks and graph signal processing literatures, namely, community detection Fortunato (2010) (we will mostly revisit the results in Guédon and Vershynin (2016) and Fei and Chen (2019) from our perspective), signed clustering Cucuringu et al. (a), group synchronization Singer (2011) and MAX-CUT.

The proof of Theorem 1 is straightforward (mostly because the loss function is linear). Its importance stems from the fact that it puts forward two important concepts originally introduced in Learning Theory, namely that the complexity of the problem comes from the one of the local subset $\mathcal{C} \cap \{Z : \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r^*(\Delta)\}$ and that the "radius" $r^*(\Delta)$ of

the localization is solution of a fixed point equation. For a setup given by a random matrix A and a constraint $\mathcal{C}$, we should try to understand how these two ideas apply to obtain estimation properties of SDP estimators such as $\hat{Z}$. That is to understand the shape of the local subsets $\mathcal{C} \cap \{Z : \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r\}, r > 0$ and the maximal oscillations of the empirical process $Z \rightarrow \langle A - \mathbb{E}A, Z - Z^* \rangle$ indexed by these local subsets. We will consider this task in three distinct problem instances. For a detailed proof of Theorem 1 (and Theorem 2 below), we refer the reader to Appendix A.

The main conclusion of Theorem 1 is that all the information for the problem of estimating Z^* via $\hat{Z}$ is contained in the fixed point $r^*(\Delta)$. We therefore have to compute or upper bound such a fixed point. This might be difficult in great generality but there are some tools that can help to find upper bounds on $r^*(\Delta)$.

A first approach is to understand the shape of the local sets $\mathcal{C} \cap \{Z : \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r\}, r > 0$ and to that end it is helpful to characterize the *curvature* of the excess risk $Z \rightarrow \langle \mathbb{E}A, Z^* - Z \rangle$ around its maximizer Z^*. This type of local characterization of the excess risk is also a tool used in Learning theory that goes back to classical conditions such as the Margin assumption Tsybakov (2004); Mammen and Tsybakov (1999) or the Bernstein condition Bartlett and Mendelson (2006). The latter condition was initially introduced as an upper bound of the variance term by its expectation: for all $Z \in \mathcal{C}$, $\mathbb{E}(\ell_A(Z) - \ell_A(Z^*))^2 \leqslant c_0 \mathbb{E}(\ell_A(Z) - \ell_A(Z^*))$ for some absolute constant c_0, but it has now been better understood as a way to discriminate the oracle from the other points in the model $\mathcal{C}$. These assumptions were *global* assumption in the sense that they concern all Z in $\mathcal{C}$. It has been recently shown Chinot et al. (2018) that only the *local curvature* of the excess risk needs to be understood. We now introduce this tool in our setup.

We characterize the local curvature of the excess risk by some function $G : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$. Most of the time the G function is a norm like the ℓ_1-norm or a power of a norm such as the ℓ_2 norm to the square. The radius defining the local subset onto which we need to understand the curvature of the excess risk is also solution of a fixed point equation:

$$r_G^*(\Delta) = \inf \left(r > 0 : \mathbb{P}\left[\sup_{Z \in \mathcal{C} : G(Z^* - Z) \leqslant r} \langle A - \mathbb{E}A, Z - Z^* \rangle \leqslant (1/2)r \right] \geqslant 1 - \Delta \right). \quad (5)$$

The difference between the two fixed points $r^*(\Delta)$ and $r_G^*(\Delta)$ is that the local subsets are not defined using the same proximity function to the oracle Z^*; the first one uses the excess risk as a proximity function while the second one uses the G function as a proximity function. The G function should play the role of a *simple* description of the curvature of the excess risk function locally around Z^*; that is formalized in the next assumption.

Assumption 1 *For all $Z \in \mathcal{C}$, if $\langle \mathbb{E}A, Z^* - Z \rangle \leqslant r_G^*(\Delta)$ then $\langle \mathbb{E}A, Z^* - Z \rangle \geqslant G(Z^* - Z)$.*

Typical examples of curvature functions G will have the form $G(Z^* - Z) = \theta \|Z^* - Z\|^\kappa$ for some $\kappa \geqslant 1$, $\theta > 0$ and some norm $\|\cdot\|$. In that case, the parameter κ was initially called the *margin parameter* Tsybakov (2003); Mammen and Tsybakov (1999). Even though the relation given in Assumption 1 has been typically referred to as a margin condition or Bernstein condition in the Learning Theory literature, we will rather call it a *local curvature assumption*, following Guédon and Vershynin (2016) and Chinot et al. (2018), since this type of relation describes the behavior of the risk function locally around its

oracle. The main advantage for finding a local curvature function G is that $r_G^*(\Delta)$ should be easier to compute than $r^*(\Delta)$ and $r^*(\Delta) \leqslant r_G^*(\Delta)$ because of the definition of $r_G^*(\Delta)$ and $\{Z \in \mathcal{C} : \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r_G^*(\Delta)\} \subset \{Z \in \mathcal{C} : G(Z^* - Z) \leqslant r_G^*(\Delta)\}$ (thanks to Assumption 1). We can therefore state the following corollary.

Corollary 1 *We assume that the constraint $\mathcal{C}$ is star-shaped in Z^* and that the "local curvature" Assumption 1 holds for some $0 < \Delta < 1$. With probability at least $1 - \Delta$, it holds true that*

$$r_G^*(\Delta) \geqslant \langle \mathbb{E}A, Z^* - \hat{Z} \rangle \geqslant G(Z^* - \hat{Z}).$$

When it is possible to describe the local curvature of the excess risk around its oracle by some G function and when some estimate of $r_G^*(\Delta)$ can be obtained, Corollary 1 applies and estimation results of Z^* by $\hat{Z}$ (w.r.t. to both the "excess risk" metric $\langle \mathbb{E}A, Z^* - \hat{Z} \rangle$ and the G metric) follow. If not, either because understanding the local curvature of the excess risk or the computation of $r_G^*(\Delta)$ is difficult, it is still possible to apply Theorem 1 with the *global* VC approach which boils down to simply upper bound the fixed point $r^*(\Delta)$ used in Theorem 1 by a global parameter that is a complexity measure of the entire set $\mathcal{C}$:

$$r^*(\Delta) \leqslant \inf \left(r > 0 : \mathbb{P}\left[\sup_{Z \in \mathcal{C}} \langle A - \mathbb{E}A, Z - Z^* \rangle \leqslant (1/2)r \right] \geqslant 1 - \Delta \right). \qquad (6)$$

Interestingly, if the latter last resort approach is used then, following the approach form Guédon and Vershynin (2016), Grothendieck's inequality Grothendieck (1953); Pisier (2012) appears to be a powerful tool to upper bound the right-hand side of (6) in the case of the community detection problem such as in Guédon and Vershynin (2016) as well as in the MAX-CUT problem. Of course, when it is possible to avoid this ultimate global, one should do so because the local approach will always provide better results.

Finally, proving a "local curvature" property such as in Assumption 1 may be difficult because it requires to understand the shape of the local subsets $\mathcal{C} \cap \{Z : \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r\}, r > 0$. It is however possible to simplify this assumption if getting estimation results of Z^* only w.r.t. the G function (and not necessarily an upper bound on the excess risk $\langle \mathbb{E}A, Z^* - \hat{Z} \rangle$) is enough. In that case, Assumption 1 may be replaced by the following one.

Assumption 2 *For all $Z \in \mathcal{C}$, if $G(Z^* - Z) \leqslant r_G^*(\Delta)$ then $\langle \mathbb{E}A, Z^* - Z \rangle \geqslant G(Z^* - Z)$.*

Assumption 2 assumes a curvature of the excess risk function in a G neighborhood of Z^* unlike Assumption 1 which grants this curvature in an "excess risk neighborhood". The shape of a neighborhood defined by the G function may be easier to understand (for instance when G is a norm, a neighborhood defined by G is the ball of a norm centered at Z^* with radius $r_G^*(\Delta)$). In general, the latter assumption and Assumption 1 do not compare. In the next result, we show that if Assumption 2 holds then $\hat{Z}$ can estimate Z^* w.r.t. the G function.

Theorem 2 *We assume that the constraint $\mathcal{C}$ is star-shaped in Z^* and that the "local curvature" Assumption 2 holds for some $0 < \Delta < 1$. We assume that the G function is continuous, $G(0) = 0$ and $G(\lambda(Z^* - Z)) \leqslant \lambda G(Z^* - Z)$ for all $\lambda \in [0, 1]$ and $Z \in \mathcal{C}$. With probability at least $1 - \Delta$, it holds true that $G(Z^* - \hat{Z}) \leqslant r_G^*(\Delta)$.*

As a consequence, Theorem 1, Corollary 1 and Theorem 2 are the three tools at our disposal to study the performance of SDP estimators depending on the deepness of understanding we have on the problem. The best approach is given by Theorem 1 when it is possible to compute efficiently the complexity fixed point $r^*(\Delta)$. If the latter approach is too complicated (likely because understanding the geometry of the local subset $\mathcal{C} \cap \{Z : \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r\}, r > 0$ may be difficult) then one may resort to find a curvature function G of the excess risk locally around Z^*. In that case, both Corollary 1 and Theorem 2 may apply depending on the hardness to find a local curvature function G on an "excess risk neighborhood" (see Assumption 1) or a "G-neighborhood" (see Assumption 2). Finally, if no local approach can be handled (likely because describing the curvature of the excess risk in any neighborhood of Z^* or controlling the maximal oscillations of the empirical process $Z \to \langle \mathbb{E}A - A, Z^* - Z \rangle$ locally are too difficult) then one may resort ultimately to a global approach which follows from Theorem 1 as explained in (6). In the following, we will use these tools for various problems.

Results like Theorem 1, Corollary 1 and Theorem 2 appeared in many papers on ERM in Learning Theory such as in Koltchinskii (2011); Bartlett and Mendelson (2006); Massart (2007); Lecué and Mendelson (2013). In all these results, typical loss functions such as the quadratic or logistic loss functions, were not linear ones, such as the one we are using here. From that point of view our problem is easier and this can be seen by the simplicity to prove our three general results from this section. What is much more complicated here than in other more classical problems in Learning Theory is the computation of the fixed point because (i) the stochastic processes $Z \to \langle A - \mathbb{E}A, Z - Z^* \rangle$ may be far from being a Gaussian process if the noise matrix $A - \mathbb{E}A$ is complicated and (ii) the local sets $\{Z \in \mathcal{C} : \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r\}$ or $\{Z \in \mathcal{C} : G(Z^* - Z) \leqslant r\}$ for $r > 0$ maybe very hard to describe in a simple way. Instrumental results are available in the literature to circumvent this kind of problems; see Fei and Chen (2019).

3. Revisiting two results from the community detection literature Fei and Chen (2019); Guédon and Vershynin (2016)

The rapid growth of social networks on the Internet has lead many statisticians and computer scientists to focus their research on data coming from graphs. One important topic that has attracted particular interest during the last decades is that of community detection Fortunato (2010); Porter et al. (2009), where the goal is to recover mesoscopic structures in a network, the so-called called communities. A community consists of group of nodes that are relatively densely connected to each other, but sparsely connected to other dense groups present within the network. The motivation for this line of work stems not only from the fact that finding communities in a network is an interesting and challenging problem of its own as it leads to understanding structural properties of networks, but community detection is also used as a data pre-processing step for other statistical inference tasks on large graphs, as it facilitates parallelization and allows one to distribute time consuming processes on several smaller subgraphs (i.e., the extracted communities).

One challenging aspect of the community detection problem arises in the setting of sparse graphs. Many of the existing algorithms, which enjoy theoretical guarantees, do so in the relatively dense regime for the edge sampling probability, where the expected average

degree is of the order $\Theta(\log n)$. The problem becomes challenging in very sparse graphs with bounded average degree. To this end, Guédon and Vershynin proposed a semidefinite relaxation for a discrete optimization problem Guédon and Vershynin (2016), an instance of which encompasses the community detection problem, and showed that it can recover a solution with any given relative accuracy even in the setting of very sparse graphs with average degree of order $O(1)$.

A subset of the existing literature for community detection and clustering relies on spectral methods, which consider the adjacency matrix associated to a graph, and employ its eigenvalues, and especially eigenvectors, in the analysis process or to propose efficient algorithms to solve the task at hand. Along these lines, Le et al. (2016) proposed a general framework for optimizing a general function of the graph adjacency matrix over discrete label assignments by projecting onto a low-dimensional subspace spanned by vectors that approximate the top eigenvectors of the expected adjacency matrix. The authors consider the problem of community detection with $k = 2$ communities, which they frame as an instance of their proposed framework, combined with a regularization step that shifts each entry in the adjacency matrix by a small constant τ, which renders their methodology applicable in the sparse regime as well.

In the remainder of this section, we focus on the community detection problem on random graphs under the general stochastic block model. We will mostly revisit the work in Guédon and Vershynin (2016) and Fei and Chen (2019) from the perspective given by Theorem 1. Indeed, thanks to this theorem, it is possible to simplify the proof of Fei and Chen (2019), by avoiding both the peeling argument and the use of the bound from Guédon and Vershynin (2016).

We first recall the definition of the generalized stochastic block model (SBM). We consider a set of vertices $V = \{1, \cdots, n\}$, and assume it is partitioned into K communities $\mathcal{C}_1, \cdots, \mathcal{C}_K$ of arbitrary sizes $|\mathcal{C}_1| = l_1, \cdots, |\mathcal{C}_K| = l_K$.

Definition 3 *For any pair of nodes $i, j \in V$, we denote by $i \sim j$ when i and j belong to the same community (i.e., there exists $k \in \{1, \ldots, K\}$) such that $i, j \in \mathcal{C}_k$), and we denote by $i \nsim j$ if i and j do not belong to the same community.*

For each pair (i, j) of nodes from V, we draw an edge between i and j with a fixed probability p_{ij} independently from the other edges. We assume that there exist numbers p and q satisfying $0 < q < p < 1$, such that

$$\begin{cases} p_{ij} > p, \text{ if } i \sim j \text{ and } i \neq j, \\ p_{ij} = 1, \text{ if } i = j, \\ p_{ij} < q, \text{ otherwise.} \end{cases} \tag{7}$$

We denote by $A = (A_{i,j})_{1 \leqslant i,j \leqslant n}$ the observed symmetric adjacency matrix, such that, for all $1 \leqslant i \leqslant j \leqslant n$, A_{ij} is distributed according to a Bernoulli of parameter p_{ij}. The community structure of such a graph is captured by the membership matrix $\bar{Z} \in \mathbb{R}^{n \times n}$, defined by $\bar{Z}_{ij} = 1$ if $i \sim j$, and $\bar{Z}_{ij} = 0$ otherwise. The main goal in community detection is to reconstruct $\bar{Z}$ from the observation A.

Spectral methods for community detection are very popular in the literature Guédon and Vershynin (2016); Fei and Chen (2019); Vershynin (2018); Blondel et al. (2008); Clauset et al. (2004). There are many ways to introduce such methods, one of which being via convex

relaxations of certain graph cut problems aiming to minimize a modularity function such as the RatioCut Newman (2006). Such relaxations often lead to SDP estimators, such as the ones introduced in Section 1.

Considering a random graph distributed according to the generalized stochastic block model, and its associated adjacency matrix A (i.e. $A = A^\top$ and $A_{ij} \sim \text{Bern}(p_{ij})$ for $1 \leqslant i \leqslant j \leqslant n$ and p_{ij} as defined in (7)), we will estimate its membership matrix $\bar{Z}$ via the following SDP estimator

$$\hat{Z} \in \underset{Z \in \mathcal{C}}{\operatorname{argmax}} \langle A, Z \rangle,$$

where $\mathcal{C} = \{Z \in \mathbb{R}^{n \times n}, Z \succeq 0, Z \geqslant 0, \text{diag}(Z) \preceq I_n, \sum_{i,j=1}^{n} Z_{ij} \leqslant \lambda\}$ and $\lambda = \sum_{i,j=1}^{n} \bar{Z}_{ij} = \sum_{k=1}^{K} |\mathcal{C}_k|^2$ denotes the number of nonzero elements in the membership matrix $\bar{Z}$. The motivation for this approach stems from the fact that the membership matrix $\bar{Z}$ is actually the oracle, i.e., $Z^* = \bar{Z}$ (see Lemma 7.1 in Guédon and Vershynin (2016) or Lemma 1 below), where

$$Z^* \in \underset{Z \in \mathcal{C}}{\operatorname{argmax}} \langle \mathbb{E}A, Z \rangle.$$

Following the strategy from Theorem 1 and from our point of view, the upper bound on $r^*(\Delta)$ from Guédon and Vershynin (2016) is the one that is based on the global approach – that is, without localization. Indeed, Guédon and Vershynin (2016) uses the observation that, for all $r > 0$, it holds true that

$$\sup_{Z \in \mathcal{C}: \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r} \langle A - \mathbb{E}A, Z - Z^* \rangle \overset{(a)}{\leqslant} \sup_{Z \in \mathcal{C}} \langle A - \mathbb{E}A, Z - Z^* \rangle \overset{(b)}{\leqslant} 2K_G \| A - \mathbb{E}A \|_{\text{cut}}, \quad (8)$$

where $\|\cdot\|_{\text{cut}}$ is the cut-norm[1] and K_G is the Grothendieck constant (Grothendieck's inequality is used in (b), see Pisier (2012); Vershynin (2018)). Therefore, the localization around the oracle Z^* by the excess risk "band" $B_r^* := \{Z : \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r\}$ is simply removed in inequality (a). As a consequence, the resulting statistical bound is based on the complexity of the entire class $\mathcal{C}$ whereas, in a localized approach, only the complexity of $\mathcal{C} \cap B_r^*$ matters. Next step in the proof of Guédon and Vershynin (2016) is a high-probability upper bound on $\|A - \mathbb{E}A\|_{\text{cut}}$ which follows from Bernstein's inequality and a union bound since one has $\|A - \mathbb{E}A\|_{\text{cut}} = \max_{x,y \in \{-1,1\}^n} \langle A - \mathbb{E}A, xy^\top \rangle$, then for all $t > 0$, $\|A - \mathbb{E}A\|_{\text{cut}} \leqslant tn(n-1)/2$ with probability at least $1 - \exp\left(2n \log 2 - (n(n-1)t^2)/(16\bar{p} + 8t/3)\right)$ where $\bar{p} \overset{def}{=} 2/[n(n-1)] \sum_{i<j} p_{ij}(1 - p_{ij})$. The resulting upper bound on the fixed point obtained in Guédon and Vershynin (2016) is,

$$r^*(\Delta) \leqslant (8/3)K_G(2n \log(2) + \log(1/\Delta)). \quad (9)$$

Finally, under the assumption of Theorem 1 in Guédon and Vershynin (2016) (i.e., for some some $\epsilon \in (0,1)$, $n \geqslant 5.10^4/\epsilon^2$, $\max(p(1-p), q(1-q)) \geqslant 20/n$, $p = a/n > b/n = q$ and $(a-b)^2 \geqslant 2.10^4 \epsilon^{-2}(a+b)$), for $\Delta = e^3 5^{-n}$ we obtain (using the general result in Theorem 1)

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1. The cut-norm $\|\cdot\|_{\text{cut}}$ of a real matrix $A = (a_{ij})_{i \in R, j \in C}$ with a set of rows indexed by R and a set of columns indexed by C, is the maximum, over all $I \subset R$ and $J \subset C$, of the quantity $|\sum_{i \in I, j \in J} a_{ij}|$. It is also the operator norm of A from ℓ_∞ to ℓ_1 and the "injective norm" in the orginal Grothendieck "résumé" Grothendieck (1956); Pisier (2012)

with probability at least $1 - \Delta$, the bound $\langle \mathbb{E}A, Z^* - \hat{Z} \rangle \leqslant r^*(\Delta) \leqslant \epsilon n^2 = \epsilon \| Z^* \|_2^2$, which is the result from Theorem 1 in Guédon and Vershynin (2016). Finally, Guédon and Vershynin (2016) uses a (global) curvature property of the excess risk in its Lemma 7.2:

Lemma 1 (Lemma 7.2 in Guédon and Vershynin (2016)) *For all $Z \in \mathcal{C}$, $\langle \mathbb{E}A, Z^* - Z \rangle \geqslant [(p - q)/2] \| Z^* - Z \|_1$.*

Therefore, a (global– that is for all $Z \in \mathcal{C}$) curvature assumption holds for a G function which is here the $\ell_1^{n \times n}$ norm, a margin parameter $\kappa = 1$ and $\theta = (p - q)/2$ for the community detection problem. However, this curvature property is not used to compute a "better" fixed point parameter but only to obtain a $\ell_1^{n \times n}$ estimation bound since

$$\left\| \hat{Z} - Z^* \right\|_1 \leqslant \left(\frac{2}{p - q} \right) \langle \mathbb{E}A, Z^* - \hat{Z} \rangle \leqslant \frac{16 K_G (2n \log(2) + \log(1/\Delta))}{3(p - q)}.$$

The latter bound together with the sin-Theta theorem allow the authors of Guédon and Vershynin (2016) to obtain estimation bound for the community membership vector x^*.

The approach from Fei and Chen (2019) improves upon the one in Guédon and Vershynin (2016) because it uses a localization argument: the curvature property of the excess risk function from Lemma 1 is used to improve the upper bound in (9) obtained following a global approach. Indeed, the authors from Fei and Chen (2019) obtain high-probability upper bound on the quantity

$$\sup_{Z \in \mathcal{C} : \| Z^* - Z \|_1 \leqslant r} \langle A - \mathbb{E}A, Z - Z^* \rangle$$

depending on r. This yields to exact reconstruction result in the "dense" case and exponentially decaying rates of convergence in the "sparse" case. This is a typical example where the localization argument shows its advantage upon the global approach. The price to pay is usually a more technical proof for the local approach compare with the global one. However, the argument from Fei and Chen (2019) also uses an unnecessary peeling argument together with an unnecessary a priori upper bound on $\left\| \hat{Z} - Z^* \right\|_1$ (which is actually the one from Guédon and Vershynin (2016)). It appears that this peeling argument and this a priori upper bound on $\left\| \hat{Z} - Z^* \right\|_1$ can be avoided thanks to our approach from Theorem 1. This improves the probability estimate and simplifies the proofs (since the result from Guédon and Vershynin (2016) is not required anymore neither is the peeling argument). For the sign clustering problem we consider below as an application of our main results, we will mostly adapt the probabilistic tools from Fei and Chen (2019) (in the "dense" case) to the methodology associated with Theorem 1 (without this two unnecessary arguments).

4. Contributions of the paper

This section encompasses the main contributions of our paper for the three problems we study, namely signed clustering, angular synchronization, and MAX-CUT.

4.1 Application to signed clustering

Much of the clustering literature, including both spectral and non-spectral methods, has focused on unsigned graphs, where each edge carries a non-negative scalar weight that encodes a measure of affinity (similarity, trust) between pairs of nodes. However, in numerous instances, the above-mentioned affinity takes negative values, and encodes a measure of dissimilarity or distrust. Such applications arise in social networks where users relationships denote trust-distrust or friendship-enmity, shopping bipartite networks which capture like-dislike relationships between users and products Banerjee et al. (2012), online news and review websites, such as Epinions and Slashdot, that allow users to approve or denounce others Leskovec et al. (2010a), and clustering financial or economic time series data Aghabozorgi et al. (2015). Such applications have spurred interest in the analysis of signed networks, which has recently become an increasingly important research topic Leskovec et al. (2010b), with relevant lines of work in the context of clustering signed networks including, in chronological order, Kunegis et al. (2010); Chiang et al. (2012); Cucuringu et al. (a, 2020).

The second application of our proposed methodology is an extension of the community detection and clustering problem to the setting of signed graphs, where, for simplicity, we assume that an edge connecting two nodes can take either -1 or $+1$ values.

4.1.1 A SIGNED STOCHASTIC BLOCK MODEL (SSBM)

We focus on the problem of clustering a K-weakly balanced graphs[2]. We consider a signed stochastic block model (SSBM) similar to the one introduced in Cucuringu et al. (a), where we are given a graph G with n nodes $\{1, \ldots, n\}$ which are divided into K communities, $\{\mathcal{C}_1, \cdots, \mathcal{C}_K\}$, such that, in the noiseless setting, edges within each community are positive and edges between communities are negative.

The only information available to the user is given by a $n \times n$ sparse adjacency matrix A constructed as follows: A is symmetric, with $A_{ii} = 1$ for all $i = 1, \ldots, n$, and for all $1 \leqslant i < j \leqslant n$, $A_{ij} = s_{ij}(2B_{ij} - 1)$ where

$$B_{ij} \sim \left\{ \begin{array}{l} \text{Bern}(p) \text{ if } i \sim j \\ \text{Bern}(q) \text{ if } i \nsim j \end{array} \right. \quad \text{and} \quad s_{ij} \sim \text{Bern}(\delta),$$

for some $0 \leqslant q < 1/2 < p \leqslant 1$ and $\delta \in (0,1)$. Moreover, all the variables B_{ij}, s_{ij} for $1 \leqslant i < j \leqslant n$ are independent.

We remark that this SSBM model is similar to the one considered in Cucuringu et al. (a), which was governed by two parameters, the sampling probability δ as above, and the noise level η, which may flip entries of the adjacency matrix.

Our aim is to recover the community membership matrix or cluster matrix $\bar{Z} = (\bar{Z}_{ij})_{i,j \leqslant n}$, with $\bar{Z}_{ij} = 1$ when $i \sim j$ and $\bar{Z}_{ij} = 0$ when $i \nsim j$ using only the observed censored adjacency matrix A.

Our approach is similar in nature to the one used by spectral methods in community detection. We first observe that for $\alpha := \delta(p + q - 1)$ and $J = (1)_{n \times n}$ we have $\bar{Z} = Z^*$

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2. A signed graph is K-weakly balanced if and only if all the edges are positive, or the nodes can be partitioned into $K \in \mathbb{N}$ disjoint sets such that positive edges exist only within clusters, and negative edges are only present across clusters Davis (1967).

where

$$Z^* \in \underset{Z \in \mathcal{C}}{\operatorname{argmax}} \langle \mathbb{E}A - \alpha J, Z \rangle \tag{10}$$

and $\mathcal{C} = \{Z \in \mathbb{R}^{n \times n} : Z \geq 0, Z_{ij} \in [0,1], Z_{ii} = 1, i = 1, \ldots, n\}$. The proof of (10) is recalled in Section B.

Since we do not know $\mathbb{E}A$ and α, we should estimate both of them. We will estimate $\mathbb{E}A$ with A but, for simplicity, we will assume that α is known. The resulting estimator of the cluster matrix $\bar{Z}$ is

$$\hat{Z} \in \underset{Z \in \mathcal{C}}{\operatorname{argmax}} \langle A - \alpha J, Z \rangle \tag{11}$$

which is indeed a SDP estimator and therefore Theorem 1 (or Corollary 1 and Theorem 2) may be used to obtain statistical bounds for the estimation of Z^* from (10) by $\hat{Z}$.

We will use the following notations: $s := \delta(p-q)^2$, $\theta := \delta(p-q)$, $\rho := \delta \max\{1 - \delta(2p-1)^2, 1 - \delta(2q-1)^2\}$, $\nu := \max\{2p-1, 1-2q\}$, $[m] := \{1, \cdots, m\}$ for all $m \in \mathbb{N}$, $l_k := |\mathcal{C}_k|$ for all $k \in [K]$, $\lambda^2 := \sum_{k=1}^{K} l_k^2$, $\mathcal{C}^+ := \overset{K}{\underset{k=1}{\cup}} (\mathcal{C}_k \times \mathcal{C}_k)$ and $\mathcal{C}^- := \underset{k \neq k'}{\cup} (\mathcal{C}_k \times \mathcal{C}_{k'})$. We also use the notation $c_0, c_1, \ldots$, to denote absolute constants whose values may change from one line to another.

4.1.2 MAIN RESULT FOR THE ESTIMATION OF THE CLUSTER MATRIX IN SIGNED CLUSTERING

Our main result concerns the reconstitution of the K communities from the observation of the matrix A. In order to avoid solutions with some communities of degenerated size (too small or too large) we consider the following assumption.

Assumption 3 *Up to constants, the elements of the partition $\mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_K$ of $\{1, \ldots, n\}$ are of same size: there are absolute constant $c_0, c_1 > 0$ such that for any $k \in [K]$, $n/(c_1 K) \leq |\mathcal{C}_k| = l_k \leq c_0 n/K$.*

We are now ready to state the main result on the estimation of the cluster matrix Z^* from (10) by the SDP estimator $\hat{Z}$ from (11).

Theorem 3 *There is an absolute positive constant c_0 such that the following holds. Grant Assumption 3. Assume that*

$$n\nu\delta \geq \log n, \tag{12}$$

$$sn \geq c_0 K^2 \nu \tag{13}$$

$$and \qquad \frac{K \log(2eKn)}{n} \leq \max\left(\frac{\theta^2}{\rho}, \frac{9\rho}{32}\right). \tag{14}$$

Then, with probability at least $1 - \exp(-\delta\nu n) - 3/(2eKn)$, exact recovery holds true, i.e., $\hat{Z} = Z^*$. We recall the constants defined above : $s := \delta(p-q)^2$, $\theta := \delta(p-q)$, $\rho := \delta \max\{1 - \delta(2p-1)^2, 1 - \delta(2q-1)^2\}$, $\nu := \max\{2p-1, 1-2q\}$.

Therefore, we have exact reconstruction in the dense case (that is under assumption (12)), which stems from condition (14). The latter condition is in the same spirit as the one in Theorem 1 of Fei and Chen (2019), it measures the SNR (signal-to-noise ratio) of the model which captures the hardness of the SSBM. As mentioned in Fei and Chen (2019), it is related to the Kesten-Stigum threshold Mossel et al. (2015). The last condition (14) basically requires that the number of clusters K is at most $n/\log n$. If this condition is dropped out, then we do not have anymore exact reconstruction but only a certified exponential rate of convergence: there exists a universal constant C_2 such that, with probability at least $1 - \exp(-\delta\nu n) - 3/(2eKn)$,

$$\left\| Z^* - \hat{Z} \right\|_1 \leqslant \frac{2en^2}{c_1\theta}\exp\left(-\frac{sn}{C_2 K}\right). \tag{15}$$

A proof of (15) can be found in Section 5.

This shows that there is a phase transition in the dense case: exact reconstruction is possible when $K \lesssim n/\log n$ and, otherwise, when $K \gtrsim n/\log n$ we only have a control of the estimation error with an exponential convergence rate.

We then get results of the same nature as in Fei and Chen (2019), or in the more recent paper Xu et al. (2020). In those two articles, the authors show the existence of a phase transition, with exact recovery in the regime $K \lesssim n/log(n)$, and exponential rate with exponent $\simeq -sn/K$ otherwise, where s is some measurement of the signal/noise ratio of the problem. Note that the estimation bound is given with respect to the $l_1^{n\times n}$ norm. This is not a surprise since it is the behaviour of the excess risk over $\mathcal{C}$ and Z^*.

4.2 Application to angular group synchronization

In this section, we introduce the group synchronization problem as well as a stochastic model for this problem. We consider a SDP relaxation of the original problem (which is exact) and construct the associated SDP estimator such as in (3).

The angular synchronization problem consists of estimating n unknown angles $\theta_1, \cdots, \theta_n$ (up to a global shift angle) given a noisy subset of their pairwise offsets $(\theta_i - \theta_j)[2\pi]$, where $[2\pi]$ is the modulo 2π operation. The pairwise measurements can be realized as the edge set of a graph G, typically modeled as an Erdös-Renyi random graph Singer (2011).

The aim of this section is to show that the angular synchronization problem can be analyzed using our methodology. In order to keep the presentation as simple as possible, we assume that all pairwise offsets are observed up to some Gaussian noise: we are given $\delta_{ij} = (\theta_i - \theta_j + \sigma g_{ij})[2\pi]$ for all $1 \leqslant i < j \leqslant n$ where $(g_{ij} : 1 \leqslant i < j \leqslant n)$ are $n(n-1)/2$ i.i.d. standard Gaussian variables and $\sigma > 0$ is the noise variance. We may rewrite the problem as follows: we observe a $n \times n$ complex matrix A defined by

$$A = S \circ [x^*(\overline{x^*})^\top] \text{ where } S = (S_{ij})_{n\times n}, S_{ij} = \begin{cases} e^{\iota\sigma g_{ij}} & \text{if } i < j \\ 1 & \text{if } i = j \\ e^{-\iota\sigma gij} & \text{if } i > j \end{cases}, \tag{16}$$

ι denotes the imaginary number such that $\iota^2 = -1$, $x^* = (x_i^*)_{i=1}^n \in \mathbb{C}^n$, $x_i^* = e^{\iota\theta_i}, i = 1,\ldots,n$, $\bar{x}$ denotes the conjugate vector of x and $S \circ [x^*(\overline{x^*})^\top]$ is the element-wise product $(S_{ij}x_i\bar{x}_j)_{n\times n}$. In particular, S is a Hermitian matrix (i.e. $\bar{S}^\top = S$) and $\mathbb{E}S_{ij} = \exp(-\sigma^2/2)$

for $i \neq j$ and $\mathbb{E}S_{ii} = 1$ if $i = j$. We want to estimate $(\theta_1, \ldots, \theta_n)$ (up to a global shift) from the matrix of data A.

Unlike the statistical model introduced in Bandeira et al. (2016), the noise here is multiplicative in A. From a physical point of view, it makes more sense to consider an additive noise on the offsets i.e. we observe $(\theta_i - \theta_j + \sigma g_{ij})[2\pi]$. The noise becomes multipicative by passing to the exponential. However, to compare our methodology with the one from Bandeira et al. (2016), we also consider the model in Bandeira et al. (2016) (that is an additive noise on the matrice $Z^* = x^*(\overline{x^*})^\top$ instead of the additive noise in A). We recover similar results in this latter model than the one in Bandeira et al. (2016). For the moment, we consider the multiplicative noise and the data matrix A as introduced above in (16); we will turn to the addtive noise model from Bandeira et al. (2016) at the very end of this section in a remark.

The first step is to find an (vectorial) optimization problem which solutions are given by $(\theta_i)_{i=1}^n$ (up to global angle shift) or some bijective function of it. Estimating $(\theta_i)_{i=1}^n$ up to global angle shift is equivalent to estimating the vector $x^* = (e^{\iota\theta_i})_{i=1}^n$. The latter is, up to a global rotation of its coordinates, the unique solution of the following maximization problem

$$\underset{x\in\mathbb{C}^n:|x_i|=1}{\operatorname{argmax}} \left\{ \bar{x}^\top \, \mathbb{E}A \, x \right\} = \{(e^{\iota(\theta_i+\theta_0)})_{i=1}^n : \theta_0 \in [0, 2\pi)\}. \tag{17}$$

A proof of (17) is given in Section 6. Let us now rewrite (17) as a SDP problem. For all $x \in \mathbb{C}^n$, we have $\bar{x}^\top \mathbb{E}Ax = \operatorname{tr}(\mathbb{E}AX) = \langle \mathbb{E}A, X \rangle$ where $X = x\bar{x}^\top$ and $\{Z \in \mathbb{C}^{n\times n} : Z = x\bar{x}^T, |x_i| = 1\} = \{Z \in \mathbb{H}_n : Z \geq 0, \operatorname{diag}(Z) = \mathbf{1}_n, \operatorname{rank}(Z) = 1\}$ where $\mathbb{H}_n$ is the set of all $n \times n$ Hermitian matrices and $\mathbf{1}_n \in \mathbb{C}^n$ is the vector with all coordinates equal to 1. It is therefore straightforward to construct a SDP relaxation of (17) by dropping the rank constraint. It appears that this relaxation is exact since, for $\mathcal{C} = \{Z \in \mathbb{H}_n : Z \geq 0, \operatorname{diag}(Z) = \mathbf{1}_n\}$,

$$\underset{Z\in\mathcal{C}}{\operatorname{argmax}}\langle \mathbb{E}A, Z \rangle = \{Z^*\}, \tag{18}$$

where $Z^* = x^*(\overline{x^*})^\top$. A proof of (18) can be found in Section 6. Finally, as we only observe A, we consider the following SDP estimator of Z^*

$$\hat{Z} \in \underset{Z\in\mathcal{C}}{\operatorname{argmax}}\langle A, Z \rangle. \tag{19}$$

In the next section, we use the strategy from Corollary 1 to obtain statistical guarantees for the estimation of Z^* by $\hat{Z}$.

Intuitively, the above maximization problem (18) attempts to preserve the given angle offsets as best as possible, by aiming to maximize the following objective function

$$\underset{\theta_1,\ldots,\theta_n\in[0,2\pi)}{\operatorname{argmax}} \sum_{i,j=1}^n e^{-\iota\theta_i} A_{ij} e^{\iota\theta_j}, \tag{20}$$

where the objective function is incremented by $+1$ whenever an assignment of angles θ_i and θ_j perfectly satisfies the given edge constraint $\delta_{ij} = (\theta_i - \theta_j)[2\pi]$ (i.e., for a *clean* edge for which $\sigma = 0$), while the contribution of an incorrect assignment (i.e., of a *very noisy* edge) will be almost uniformly distributed on the unit circle in the complex plane. Due

to non-convexity of optimization in (20), it is difficult to solve computationally Zhang and Huang (2006); one way to overcome this problem is to consider the SDP relaxation from (18) such as in (19) but it is also possible to consider a spectral relaxation such as the one proposed by Singer (2011) which replaces the individual constraints that all z_i's should have unit magnitude by the much weaker single constraint $\sum_{i=1}^{n} |z_i|^2 = n$, leading to

$$\underset{z_1,\ldots,z_n \in \mathbb{C};\ \sum_{i=1}^{n} |z_i|^2 = n}{\operatorname{argmax}} \sum_{i,j=1}^{n} \bar{z}_i A_{ij} z_j. \tag{21}$$

The solution to the resulting maximization problem is simply given by a top eigenvector of the Hermitian matrix A, followed by a normalization step. We remark that the main advantage of the SDP relaxation (18) is that it explicitly imposes the unit magnitude constraint for $e^{\iota\theta_i}$, which we cannot otherwise enforce in the spectral relaxation solved via the eigenvector method in (21). The above SDP program (18) is very similar to the well-known Goemans-Williamson SDP relaxation for the seminal MAX-CUT problem of finding the maximal cut of a graph (the MAX-CUT problem is one of the four applications considered in this work, see Section 4.3 below), with the main difference being that here we optimize over the cone of complex-valued Hermitian positive semidefinite matrices, not just real symmetric matrices.

4.2.1 Main results for phase recovery in the synchronization problem (in the multiplicative noise model)

Our main result concerns the estimation of the matrix of offsets $Z^* = x^*(\overline{x^*})^\top$ from the observation of the matrix A. This result is then used to estimate (up to a global phase shift) the angular vector $x^* = (e^{-\iota\theta_i})_{i=1}^n$. Our first result follows from Corollary 1.

Theorem 4 *Let $0 < \epsilon < 1$. If $\sigma \leqslant \sqrt{\log(\epsilon n^4)}$ then, with probability at least $1 - \exp(-\epsilon\sigma^4 n(n-1)/2)$, it holds true that*

$$(e^{-\sigma^2/2}/2)\,\|Z^* - Z\|_2^2 \leqslant \langle \mathbb{E}A, Z^* - Z \rangle \leqslant (128/6)\sqrt{\epsilon}\sigma^4 n(n-1). \tag{22}$$

Once we have an estimator $\hat{Z}$ for the oracle Z^*, we can extract an estimator $\hat{x}$ for the vector of phases x^* by considering a top eigenvector (i.e. an eigenvector associated with the largest eigenvalue) of $\hat{Z}$. It is then possible to quantify the estimation properties of x^* by $\hat{x}$ using a sin-Theta theorem and Theorem 4.

Corollary 2 *Let $\hat{x}$ be a top eigenvector of $\hat{Z}$ with Euclidean norm $\|\hat{x}\|_2 = \sqrt{n}$. Let $0 < \epsilon < 1$ and assume that $\sigma \leqslant \sqrt{\log(\epsilon n^4)}$. We have the existence of a universal constant $c_0 > 0$ (which is the constant in the Davis-Kahan theorem for Hermitian matrices) such that, with probability at least $1 - \exp(-\epsilon\sigma^4 n(n-1)/2)$, it holds true that*

$$\min_{z \in \mathbb{C}: |z|=1} \|\hat{x} - zx^*\|_2 \leqslant 8c_0\sqrt{2/3}\epsilon^{1/4}e^{\sigma^2/4}\sigma^2\sqrt{n}. \tag{23}$$

It follows from Corollary 2 that we can estimate x^* (up to a global rotation $z \in \mathbb{C}$: $|z| = 1$) with a ℓ_2^n-estimation error of the order of $\sigma^2\sqrt{n}$ with exponential deviations. Given

that $\|x^*\|_2 = \sqrt{n}$, this means that a constant proportion of the entries are well estimated when ϵ is taken like a constant. For a values of $\epsilon \sim 1/n^2$, the rate of estimation is like σ^2, we therefore get a much better estimation of x^* but only with constant probability. It is important to recall that $\hat{Z}$ and $\hat{x}$ can be both efficiently computed by solving a SDP problem and then by considering a top eigenvector of its solution (for instance, using the power method).

We finish the section on the angular group synchronization with the additive model as considered in Bandeira et al. (2016). Our aim is still to put forward our methodology and to show that it has a wide spectrum of applications and that, in particular, it covers also the model introduced in Bandeira et al. (2016).

4.2.2 The angular group synchronization model with additive noise from Bandeira et al. (2016)

As mentioned above, we chose to study a multiplicative noise model in A since it makes more sense from a physical point of view to have an additive noise on the offsets $\delta_{ij} = \theta_i - \theta_j[2\pi]$ (this additive noise becoming multiplicative by passing to the exponential in A). However, in Bandeira et al. (2016), the authors considered a model with additive noise on $Z^* = x^*(\overline{x^*})^\top$. In this "additive" model, we observe $C = Z^* + \sigma W$, where W is a complex Wigner matrix and $\sigma > 0$ is the noise level. The MLE $\tilde{x}$ is solution to the problem

$$\tilde{x} \in \underset{x \in \mathbb{C}^n, |x_i|=1 \forall i}{\operatorname{argmax}} \; x^T C x \tag{24}$$

which can be hard to compute in practice. Using the same approach as above, a SDP relaxation can be obtained by removing a rank one constraint, this yields to the SDP estimator

$$\tilde{Z} \in \underset{Z \in \mathcal{C}}{\operatorname{argmax}} \langle C, X \rangle \tag{25}$$

where $\mathcal{C} := \{Z \in \mathbb{H}_n : \operatorname{diag}(Z) = 1_n \text{ and } Z \geq 0\}$. Statistical properties of $\tilde{Z}$ have been obtained in Bandeira et al. (2016). We recall this result now.

Theorem 5 (Theorem 2.1 in Bandeira et al. (2016)) *Let $\tilde{x}$ be a solution of (24). Then with probability at least $1 - \mathcal{O}(n^{-\frac{5}{4}})$, $\min_{z \in \mathbb{C}: |z|=1} \|z\tilde{x} - x^*\|_2 \leq 12\sigma$. Moreover, if $\sigma \leq (1/18)n^{1/4}$, then (25) has a unique solution which is the rank one matrix $\tilde{x}\tilde{x}^\top$.*

Our methodology (here Corollary 1 is applied) may also be used to handle the "additive" noise model from Bandeira et al. (2016). We consider the same SDP estimator $\tilde{Z}$ as defined in (25) and we obtain the following result.

Theorem 6 *Let $\bar{x}$ be a top eigenvector of $\tilde{Z}$. With probability at least $1 - 5\exp(-n/2)$,*

$$\min_{z \in \mathbb{C}: |z|=1} \|z\bar{x} - x^*\|_2 \leq 40c_0\sigma$$

where c_0 is the constant appearing in the Davis-Kahan theorem for Hermitian matrices.

Compare with Theorem 2.1 from Bandeira et al. (2016), the estimation rate that we get for estimator $\bar{x}$ is the same (up to an absolute constant) as the one obtained for the MLE $\tilde{x}$ in Theorem 5, it is of the order of σ. Note however that our result for $\bar{x}$ holds without any restriction of the noise level σ whereas in Theorem 5 one needs $\sigma \leqslant (1/18)n^{1/4}$ to get this result for $\bar{x}$. Note also that our result holds with exponentially large probability whereas the one in Theorem 5 holds only with polynomial deviation. From a statistical perspective our result improves the one from Bandeira et al. (2016). However, the main interest of Theorem 5 is not on the statistical performance of $\bar{x}$ but on the sharpness of the SDP relaxation since it shows that the SDP relaxation (25) is actually exact when $\sigma \leqslant (1/18)n^{1/4}$. This is a result that we do not have and that our methodology cannot obtain since it is designed to prove only an estimation bound. But from a statistical point of view, it does not improve the estimation rate to know that the SDP relaxation is exact: our result shows that SDP relaxation is doing as good as MLE without proving that they are the same (up to global phase).

A proof of Theorem 6 is given in Annex C. We actually provide three estimation bounds for $\bar{x}$ in this proof. We are doing so because our aim is to show how a general methodology works in various examples. This methodology relies on the computation of a fixed point ($r_G^*(\Delta)$ from (5) here). Hence, understanding how to bound this fixed point is part of the objective of this paper. We therefore use the angular group synchronization problem with additive noise as a playground to show three different ways to upper bound such a fixed point. Using the three computations we actually obtained the following three upper bounds

$$\min_{z \in \mathbb{C}: |z|=1} \|z\bar{x} - x^*\|_2 \leqslant \begin{cases} 8c_0\sigma\sqrt{n} & \text{with probability at least } 1 - \exp(-n^2/2) \\ c_0\sqrt{36K_G^{\mathbb{C}}\sigma}n^{1/4} & \text{with probability at least } 1 - \exp(-n/2) \\ 40c_0\sigma & \text{with probability at least } 1 - 5\exp(-n/2) \end{cases}$$

where c_0 is the constant appearing in the Davis-Kahan theorem for Hermitian matrices and $K_G^{\mathbb{C}}$ is Grothendieck constant in the complex case. Each of the three bounds above follows from different upper bounds on the complexity fixed point r_G^*; for instance, the second one follows from the 'global' approach and the third one follows from a decomposition similar to the one from Fei and Chen (2019) and is the one we used in Theorem 6.

4.3 Application to the MAX-CUT problem

Let $A^0 \in \{0,1\}^{n \times n}$ be the adjacency (symmetric) matrix of an undirected graph $G = (V, E^0)$, where $V := \{1, \ldots, n\}$ is the set of the vertices and the set of edges is $E^0 := E \cup E^\top \cup \{(i,i) : A_{ii}^0 = 1\}$ where $E := \{(i,j) \in V^2 \ : \ i < j \text{ and } A_{ij}^0 = 1\}$ and $E^\top = \{(j,i) : (i,j) \in E\}$. We assume that G has no self loop so that $A_{ii}^0 = 0$ for all $i \in V$. A *cut* of G is any subset S of vertices in V. For a cut $S \subset V$, we define its weight by $\mathrm{cut}(G,S) := (1/2)\sum_{(i,j) \in S \times \bar{S}} A_{ij}^0$, that is the number of edges in E between S and its complement $\bar{S} = V \backslash S$. The MAX-CUT problem is to find the cut with maximal weight:

$$S^* \in \underset{S \subset V}{\mathrm{argmax}}\, \mathrm{cut}(G,S). \tag{26}$$

The MAX-CUT problem is a NP-complete problem but Goemans and Williamson (1995) constructed a 0.878 approximating solution via a SDP relaxation. Indeed, one can

write the MAX-CUT problem in the following way. For a cut $S \subset V$, we define the membership vector $x \in \{-1, 1\}^n$ associated with S by setting $x_i := 1$ if $i \in S$ and $x_i = -1$ if $i \notin S$ for all $i \in V$. We have $\mathrm{cut}(G, S) = (1/4) \sum_{i,j=1}^n A_{ij}^0 (1 - x_i x_j) := \mathrm{cut}(G, x)$ and so solving (26) is equivalent to solving

$$x^* \in \underset{x \in \{-1,1\}^n}{\mathrm{argmax}} \, \mathrm{cut}(G, x). \tag{27}$$

Since $(x_i x_j)_{i,j} = x x^\top$, the latter problem is also equivalent to solving

$$\max \left(\frac{1}{4} \sum_{i,j=1}^n A_{ij}^0 (1 - Z_{ij}) : \mathrm{rank}(Z) = 1, Z \succeq 0, Z_{ii} = 1 \right) \tag{28}$$

which admits a SDP relaxation by removing the rank 1 constraint. This yields the following SDP relaxation problem of MAX-CUT from Goemans and Williamson (1995):

$$Z^* \in \underset{Z \in \mathcal{C}}{\mathrm{argmin}} \langle A^0, Z \rangle \tag{29}$$

where $\mathcal{C} := \{ Z \in \mathbb{R}^{n \times n} : Z \succeq 0, Z_{ii} = 1, \forall i = 1, \ldots, n \}$.

Unlike the other examples from the previous sections, the SDP relaxation in (65) is not exact, except for bipartite graphs; see Khot and Naor (2009); Gärtner and Matoušek (2012) for more details. Nevertheless, thanks to the approximation result from Goemans and Williamson (1995), we can use our methodology to estimate Z^* and then deduce an approximate optimal cut. The MAX-CUT problem is therefore a good setup for us to test our methodology in a context where the SDP relaxation is not exact, but still widely used in practice. Thus the type of question we want to answer here is: what can we say in a setup where only partial or noisy information is available on $\mathbb{E}[A]$, and when the SDP relaxation associated with $\mathbb{E}[A]$ is also not exact? This differs from the previous setup where exactness of the SDP relaxation holds, and this interesting peculiarity is one of the reasons why we have chosen to present this problem here. Our motivation stems from the observation that, in many situations, the adjacency matrix A^0 is only partially observed, but nevertheless, it might be interesting to find an approximating solution to the MAX-CUT. Let us then introduce a stochastic model for the partial information available on $\mathbb{E}[A]$, the adjacency matrix here.

We observe $A = S \circ A^0 = (s_{ij} A_{ij}^0)_{1 \leqslant i,j \leqslant n}$ a "masked" version of A^0, where $S \in \mathbb{R}^{n \times n}$ is symmetric with upper triangular matrix filled with i.i.d. Bernoulli entries: for all $i, j \in V$ such that $i \leqslant j$, $S_{ij} = S_{ji} = s_{ij}$ where $(s_{ij})_{i \leqslant j}$ is a family of i.i.d. Bernoulli random variables with parameter $p \in (1/2, 1)$. Let $B := -(1/p)A$ so that $\mathbb{E}[B] = -A^0$. We can write Z^* as an oracle since $Z^* \in \mathrm{argmax}_{Z \in \mathcal{C}} \langle \mathbb{E}B, Z \rangle$ and so we estimate Z^* via the SDP estimator $\hat{Z} \in \mathrm{argmax}_{Z \in \mathcal{C}} \langle B, Z \rangle$. Our first aim is to quantify the cost we pay by using $\hat{Z}$ instead of Z^* in our final choice of cut. It appears that the fixed point used in Theorem 1 may be used to quantify this loss

$$r^*(\Delta) = \inf \left(r > 0 : \mathbb{P} \left[\sup_{Z \in \mathcal{C} : \langle \mathbb{E}B, Z^* - Z \rangle \leqslant r} \langle B - \mathbb{E}B, Z - Z^* \rangle \leqslant (1/2)r \right] \geqslant 1 - \Delta \right). \tag{30}$$

Our second result is an explicit high-probability upper bound on the latter fixed point.

4.3.1 MAIN RESULTS FOR THE MAX-CUT PROBLEM

In this section, we gather the two results on the estimation of Z^* from $\hat{Z}$ and on the approximate optimality of the final cut constructed from $\hat{Z}$. Let us now explicitly provide the construction of this cut. We consider the same strategy as in Goemans and Williamson (1995). Assume that $\hat{Z}$ has been constructed. Let $\hat{G}$ be a centered Gaussian vector with covariance matrix $\hat{Z}$. Let $\hat{x}$ be the sign vector of $\hat{G}$. Using the statistical properties of $\hat{Z}$, it is possible to prove near optimality of $\hat{x}$.

We denote the optimal values of the MAX-CUT problem associated with the graph G and its SDP relaxation by

$$\mathrm{SDP}(G) := (1/4)\langle A^0, J - Z^*\rangle = \max_{Z \in \mathcal{C}} \frac{1}{4}\sum_{i,j} A^0_{i,j}(1 - Z_{ij}) \text{ and } \mathrm{MAXCUT}(G) := \mathrm{cut}(G, S^*)$$

where S^* is a solution of (26) and $J = (1)_{n \times n}$. Our first result is to show how the 0.878 approximating result from Goemans and Williamson (1995) is downgraded by the incomplete information we have on the graph (we only partially observed the adjacency matrix A^0 via the masked matrix A).

Theorem 7 *For all $0 < \Delta < 1$. With probability at least $1 - \Delta$ (with respect to the masked S),*

$$\mathrm{SDP}(G) \geqslant \mathbb{E}\left[\mathrm{cut}(G, \hat{x})|\hat{Z}\right] \geqslant 0.878\mathrm{SDP}(G) - \frac{0.878 r^*(\Delta)}{4}.$$

To make the notation more precise, $\hat{x}$ is the sign vector of $\hat{G}$ which is a centered Gaussian variable with covariance $\hat{Z}$. In that context, $\mathbb{E}\left[\mathrm{cut}(G, \hat{x})|\hat{Z}\right]$ is the conditional expectation according to $\hat{G}$ for a fixed $\hat{Z}$. Moreover, the probability "at least $1 - \Delta$" that we obtain is w.r.t. the random masks, that is to the randomness in A.

Let us now frame Theorem 7 into the following perspective. If we had known the entire adjacency matrix (which is the case when $p = 1$), then we could have used Z^* instead of $\hat{Z}$. In that case, for $x^\star$ the sign vector of $G^\star \sim \mathcal{N}(0, Z^*)$, we know from Goemans and Williamson (1995) that

$$\mathrm{SDP}(G) \geqslant \mathbb{E}\left[\mathrm{cut}(G, x^\star)\right] \geqslant 0.878\mathrm{SDP}(G). \tag{31}$$

Therefore, Theorem 7 characterizes the price we pay for not observing the entire adjacency matrix A^0, but only a masked version A of it. It is an interesting output of Theorem 7 to observe that the fixed point $r^*(\Delta)$ measures, in a quantitative way, this loss. If we were able to identify scenarios of p and E for which $r^*(\Delta) = 0$, that would prove that there is no loss for partially observing A^0 in the MAX-CUT problem. The approach we use to control $r^*(\Delta)$ is the global one, which does not allow for exact reconstruction (that is, to show that $r^*(\Delta) = 0$).

Let us now turn to an estimation result of Z^* by $\hat{Z}$ via an upper bound on $r^*(\Delta)$.

Theorem 8 *With probability at least $1 - 4^{-n}$:*

$$\langle \mathbb{E}B, Z^* - \hat{Z}\rangle \leqslant r^*(4^{-n}) \leqslant 2n\sqrt{\frac{(2\log 4)(1 - p)(n - 1)}{p}} + \frac{8n\log 4}{3}.$$

In particular, it follows from the approximation result from Theorem 7 and the high-probability upper bound on $r^*(\Delta)$ from Theorem 8 that, with probability at least $1 - 4^{-n}$

$$\mathbb{E}\left[\mathrm{cut}(G, \hat{x})|\hat{Z}\right] \geqslant 0.878\mathrm{SDP}(G) - \frac{0.878}{4}\left(2n\sqrt{\frac{(2\log 4)(1-p)(n-1)}{p}} + \frac{8n\log 4}{3}\right). \quad (32)$$

This result is non-trivial only when the right-hand side term is strictly larger than $0.5 \cdot \mathrm{SDP}(G)$, which is the performance of a random cut. As a consequence, (32) shows that one can still do better than randomness even in an incomplete information setup for the MAX-CUT problem when p, n and $\mathrm{SDP}(G)$ are such that

$$0.378\mathrm{SDP}(G) > \frac{0.878}{4}\left(2n\sqrt{\frac{(2\log 4)(1-p)(n-1)}{p}} + \frac{8n\log 4}{3}\right).$$

For instance, when p is like a constant, it requires $\mathrm{SDP}(G)$ to be larger than $c_0 n^{3/2}$ (for some absolute constant c_0) and when $p = 1 - 1/n$, it requires $\mathrm{SDP}(G)$ to be at least $c_0 n$ (for some absolute constant c_0).

Remark 1 *To get exact recovery, that is $r^*(\Delta) = 0$, in the MAX-CUT problem (which shows that there is no loss for the MAX-CUT problem by observing only a masked version of the adjacency matrix), we have to develop a local approoach, as for the Signed Clustering and the Group Synchronization problems. To that end, we would need to solve the following two problems: 1) Find a curvature for the objective function $Z \to \langle \mathbb{E}B, Z^* - Z\rangle$ and 2) Study the oscillations of the empirical process $Z \to \langle \mathbb{E}B - B, Z^* - Z\rangle$. We leave those two difficult problems for future research.*

5. Proof of Theorem 3 (signed clustering)

The aim of this paper is to put forward a methodology developed in Learning Theory for the study of SDP estimators. In each example, we follow this methodology. For a problem, such as the signed clustering, where it is possible to characterize the curvature of the excess risk, we start to identify this curvature because the curvature function G, coming out of it, defines the local subsets of $\mathcal{C}$ driving the complexity of the problem. Then, we turn to the stochastic part of the proof, which is entirely summarized into the complexity fixed point $r_G^*(\Delta)$ from (5). Finally, we put the two pieces together and apply the main general result from Corollary 1 to obtain estimation results for the SDP estimator (11) in the signed clustering problem, which is summarized in Theorem 3.

5.1 Curvature equation

In this section, we show that the objective function $Z \in \mathcal{C} \to \langle Z, \mathbb{E}A - \alpha J\rangle$ satisfies a curvature assumption around its maximizer Z^* with respect to the $\ell_1^{n \times n}$-norm given by $G(Z^* - Z) = \theta \|Z^* - Z\|_1$ with parameter $\theta = \delta(p - q)$ (and margin exponent $\kappa = 1$).

Proposition 1 *For $\theta = \delta(p - q)$, we have for all $Z \in \mathcal{C}$, $\langle \mathbb{E}A - \alpha J, Z^* - Z\rangle = \theta \|Z^* - Z\|_1$.*

Proof Let Z be in $\mathcal{C}$. We have

$$
\begin{aligned}
\langle Z^* - Z, \mathbb{E}A - \alpha J \rangle &= \sum_{i,j=1}^n (Z^* - Z)_{ij}(\mathbb{E}A_{ij} - \alpha) \\
&= \sum_{(i,j)\in\mathcal{C}^+} (Z_{ij}^* - Z_{ij})(\delta(2p-1) - \alpha) + \sum_{(i,j)\in\mathcal{C}^-} (Z_{ij}^* - Z_{ij})(\delta(2q-1) - \alpha) \\
&= \delta(p-q)\left[\sum_{(i,j)\in\mathcal{C}^+} (Z^* - Z)_{ij} - \sum_{(i,j)\in\mathcal{C}^-} (Z^* - Z)_{ij}\right].
\end{aligned}
$$

Moreover, for all $(i,j) \in \mathcal{C}^+$, $Z_{ij}^* = 1$ and $0 \leqslant Z_{ij} \leqslant 1$, so $(Z^* - Z)_{ij} = |(Z^* - Z)_{ij}|$. We also have for all $(i,j) \in \mathcal{C}^-$, $(Z^* - Z)_{ij} = -Z_{ij} = -|(Z^* - Z)_{ij}|$ because in that case $Z_{ij}^* = 1$ and $0 \leqslant Z_{ij} \leqslant 1$. Hence,

$$
\langle Z^* - Z, \mathbb{E}A - \alpha J \rangle = \delta(p-q)\left[\sum_{(i,j)\in\mathcal{C}^+} |(Z^* - Z)_{ij}| + \sum_{(i,j)\in\mathcal{C}^-} |(Z^* - Z)_{ij}|\right] = \theta \left\| Z^* - Z \right\|_1.
$$

$\blacksquare$

5.2 Computation of the complexity fixed point $r_G^*(\Delta)$

Define $W := A - \mathbb{E}A$ the noise matrix of the problem. Since W is symmetric, its entries are not independent. In order to work only with independent random variables, we define the following matrix $\Psi \in \mathbb{R}^{n\times n}$:

$$
\Psi_{ij} = \begin{cases} W_{ij} \text{ if } i \leqslant j \\ 0 \text{ otherwise,} \end{cases} \tag{33}
$$

where 0 entries are considered as independent Bernoulli variables with parameter 0 and therefore, Ψ has independent entries, and satisfies the relation $W = \Psi + \Psi^\top$.

In order to obtain upper bounds on the fixed point complexity parameter $r_G^*(\Delta)$ associated with the signed clustering problem, we need to prove a high-probability upper bound on the quantity

$$
\sup_{Z\in\mathcal{C}:\|Z-Z^*\|_1 \leqslant r} \langle W, Z - Z^* \rangle, \tag{34}
$$

and then find a radius r as small as possible such that the quantity in (34) is less than $(\theta/2)r$. We denote $\mathcal{C}_r := \mathcal{C} \cap (Z^* + rB_1^{n\times n}) = \{Z \in \mathcal{C} : \|Z - Z^*\|_1 \leqslant r\}$ where $B_1^{n\times n}$ is the unit $\ell_1^{n\times n}$-ball of $\mathbb{R}^{n\times n}$.

We follow the strategy from Fei and Chen (2019) by decomposing the inner product $\langle W, Z - Z^* \rangle$ into two parts according to the SVD of Z^*. This observation is a key point in the work of Fei and Chen (2019) compared to the analysis from Guédon and Vershynin (2016). This allows to perform the localization argument efficiently. Up to a change of index of the nodes, Z^* is a block matrix with K diagonal blocks of 1's. It therefore admits K singular vectors $U_{\bullet k} := I(i \in \mathcal{C}_k)/\sqrt{|\mathcal{C}_k|}$ with multiplicity l_k associated with the singular

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value l_k for all $k \in [K]$. We can therefore write

$$Z^* = \sum_{k=1}^{K} l_k U_{\bullet k} \otimes U_{\bullet k} = U D U^\top,$$

where $U \in \mathbb{R}^{n \times K}$ has K column vectors given by $U_{\bullet k}, k = 1, \ldots, K$ and $D = \mathrm{diag}(l_1, \ldots, l_K)$. We define the following projection operator

$$\mathcal{P} : M \in \mathbb{R}^{n \times n} \to U U^T M + M U U^T - U U^T M U U^T$$

and its orthogonal projection $\mathcal{P}^\perp$ by

$$\mathcal{P}^\perp : M \in \mathbb{R}^{n \times n} \to M - \mathcal{P}(M) = (\mathrm{I}_n - U U^T) M (\mathrm{I}_n - U U^T) = \sum_{k=K+1}^{n} \langle M, U_{\bullet k} \otimes U_{\bullet k} \rangle U_{\bullet k} \otimes U_{\bullet k}$$

where $U_{\bullet k} \in \mathbb{R}^n, k = K+1, \ldots, n$ are such that $(U_{\bullet k} : k = 1, \ldots, n)$ is an orthonormal basis of $\mathbb{R}^n$.

We use the same decomposition as in Fei and Chen (2019): for all $Z \in \mathcal{C}$,

$$\langle W, Z - Z^* \rangle = \langle W, \mathcal{P}(Z - Z^*) + \mathcal{P}^\perp(Z - Z^*) \rangle = \underbrace{\langle \mathcal{P}(Z - Z^*), W \rangle}_{S_1(Z)} + \underbrace{\langle \mathcal{P}^\perp(Z - Z^*), W \rangle}_{S_2(Z)}.$$

The next step is to control with large probability the two terms $S_1(Z)$ and $S_2(Z)$ uniformly for all $Z \in \mathcal{C} \cap (Z^* + r B_1^{n \times n})$. To that end we use the two following propositions where we recall that $\rho = \delta \max(1 - \delta(2p-1)^2, 1 - \delta(2q-1)^2)$ and $\nu = \max(2p-1, 1-2q)$. The proof of Proposition 2 and 3 can be found in Section B, it is based on Fei and Chen (2019).

Proposition 2 *There are absolute positive constants c_0, c_1, c_2 and c_3 such that the following holds. If $\lceil c_1 r K/n \rceil \geq 2 e K n \exp(-(9/32) n \rho / K)$ then we have*

$$\mathbb{P} \left[\sup_{Z \in \mathcal{C} \cap (Z^* + r B_1^{n \times n})} S_1(Z) \leq c_2 r \sqrt{\frac{K \rho}{n} \log \left(\frac{2 e K n}{\lceil \frac{c_1 r K}{n} \rceil} \right)} \right] \geq 1 - 3 \left(\frac{\lceil \frac{c_1 r K}{n} \rceil}{2 e K n} \right)^{\lceil \frac{c_1 r K}{n} \rceil}.$$

Proposition 3 *There exists an absolute constant $c_0 > 0$ such that the following holds. When $n \nu \delta \geq \log n$, with probability at least $1 - \exp(-\delta \nu n)$,*

$$\sup_{Z \in \mathcal{C} \cap (Z^* + r B_1^{n \times n})} S_2(Z) \leq c_0 K r \sqrt{\frac{\delta \nu}{n}}.$$

It follows from Proposition 2 and Proposition 3 that when $n \nu \delta \geq \log n$, for all r such that $\lceil c_1 r K/n \rceil \geq 2 e K n \exp(-(9/32) n \rho / K)$ we have, for

$$\Delta = \Delta(r) := \exp(-\delta \nu n) - 3 \left(\lceil \frac{c_1 r K}{n} \rceil / (2 e K n) \right)^{\lceil \frac{c_1 r K}{n} \rceil},$$

with probability at least $1 - \Delta$,

$$\sup_{Z \in \mathcal{C} \cap (Z^* + rB_1^{n \times n})} \langle W, Z - Z^* \rangle \leqslant c_0 K r \sqrt{\frac{\delta\nu}{n}} + c_2 r \sqrt{\frac{K\rho}{n} \log\left(\frac{2eKn}{\lceil \frac{c_1 rK}{n} \rceil}\right)}.$$

Moreover, we have

$$c_0 K r \sqrt{\frac{\delta\nu}{n}} + c_2 r \sqrt{\frac{K\rho}{n} \log\left(\frac{2eKn}{\lceil \frac{c_1 rK}{n} \rceil}\right)} \leqslant \frac{\theta}{2} r \qquad (35)$$

for $\theta = \delta(p - q)$ when $K\sqrt{\nu} \lesssim \sqrt{n}\delta(p - q)$ and $\lceil c_1 rK/n \rceil \geqslant 2eKn \exp(-\theta^2 n/(K\rho))$. In particular, when $(p - q)^2 n\delta \geqslant K^2 \nu$ and $1 \geqslant 2eKn \max\left(\exp(-\theta^2 n/(K\rho)), \exp(-(9/32)n\rho/K)\right)$, we conclude that for all $0 < r \leqslant n/(c_1 K)$ (35) is true. Therefore, one can take $r_G^*(\Delta(0)) = 0$ meaning that we have exact reconstruction of Z^*: if $(p - q)^2 n\delta \geqslant K^2 \nu$ and $n \gtrsim K \max(\rho/\theta^2 \log(2eK^2\rho/\theta^2), (1/\rho) \log(2eK^2/\rho))$ then with probability at least $1 - \exp(-\delta\nu n) - 3/(2eKn)$, $\hat{Z} = Z^*$.

If $(p - q)^2 n\delta \geqslant K^2 \nu$ and $1 < 2eKn \max\left(\exp(-\theta^2 n/(K\rho)), \exp(-(9/32)n\rho/K)\right)$ then we do not have exact reconstruction anymore, but we see that (35) is true for any r such that $2eKn \exp(-\theta^2 n/(16c_2^2 K\rho)) \leqslant \lceil \frac{c_1 rK}{n} \rceil \leqslant 2eKn \exp(-c_0^2 K\delta\nu/(c_2^2\rho))$, which is possible since $(p - q)^2 n\delta \geqslant K^2 \nu$, and then we can conclude that $r^*(\Delta) \leqslant \frac{2en^2}{c_1} \exp(-\theta^2 n/(16c_2^2 K\rho))$.

Therefore, it follows from Corollary 1 that

$$\left\| Z^* - \hat{Z} \right\|_1 \leqslant \frac{2en^2}{c_1\theta} \exp\left(-\frac{\theta^2 n}{16c_2^2 K\rho}\right).$$

6. Proofs of Theorem 4 and Corollary 2 (angular synchronization)

Proof of (17): We recall that the offsets are $\delta_{ij} = \theta_i - \theta_j [2\pi]$ and we will use that if g is $\mathcal{N}(0, 1)$ then $\mathbb{E}e^{\iota\sigma g} = e^{-\sigma^2/2}$. For all $\gamma_1, \ldots, \gamma_n \in [0, 2\pi)$ we have $\gamma_i - \gamma_j = \delta_{ij}$ for all $i \neq j \in [n]$ if and only if $e^{\sigma^2/2} \mathbb{E}A_{ij} e^{\iota\gamma_j} - e^{\iota\gamma_i} = 0$ for all $i \neq j \in [n]$. We therefore have

$$\operatorname*{argmin}_{x \in \mathbb{C}^n : |x_i| = 1} \left\{ \sum_{i \neq j} |e^{\sigma^2/2} \mathbb{E}A_{ij} x_j - x_i|^2 \right\} = \{(e^{\iota(\theta_i + \theta_0)})_{i=1}^n : \theta_0 \in [0, 2\pi)\}. \qquad (36)$$

Moreover, for all $x = (x_i)_{i=1}^n \in \mathbb{C}^n$ such that $|x_i| = 1$ for $i = 1, \ldots, n$, we have

$$\sum_{i \neq j} |e^{\sigma^2/2} \mathbb{E}A_{ij} x_j - x_i|^2 = \sum_{i \neq j} |x_i^* \bar{x}_j^* x_j - x_i|^2 = \sum_{i,j=1}^n |x_i^* \bar{x}_j^* - x_i \bar{x}_j|^2$$

$$= 2n^2 - 2\Re\left(\sum_{i,j=1}^n x_i^* \bar{x}_j^* x_j \bar{x}_i\right) = 2n^2 - 2|\langle x^*, x\rangle|^2$$

where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$. On the other side, we have

$$\bar{x}^\top (e^{\sigma^2/2} \mathbb{E}A)x = \sum_{i \neq j} \bar{x}_i x_i^* \bar{x}_j^* x_j + \sum_{i=1}^n \bar{x}_i e^{\sigma^2/2} x_i = n(e^{\sigma^2/2} - 1) + |\langle x^*, x\rangle|^2.$$

Hence, minimizing $x \to \sum_{i \neq j}^{n} |e^{\sigma^2/2}\mathbb{E}A_{ij}x_j - x_i|^2$ over all $x = (x_i)_i \in \mathbb{C}^n$ such that $|x_i| = 1$ is equivalent to maximize $x \to \bar{x}^\top \mathbb{E}Ax$ over all $x = (x_i)_i \in \mathbb{C}^n$ such that $|x_i| = 1$. This concludes the proof with (36). ∎

Proof of (18)**:** Let $\mathcal{C}' = \{Z \in \mathbb{C}^{n \times n} : |Z_{ij}| \leqslant 1, \forall i, j \in [n]\}$. We first prove that $\mathcal{C} \subset \mathcal{C}'$. Let $Z \in \mathcal{C}$. Since $Z \succeq 0$, there exists $X \in \mathbb{C}^{n \times n}$ such that $Z = X\bar{X}^\top$. For all $i \in \{1, \ldots, n\}$, denote by $X_{i\bullet}$ the i-th row vector of X. We have $\|X_{i\bullet}\|_2^2 = \langle X_{i\bullet}, X_{i\bullet} \rangle = Z_{ii} = 1$ since $\mathrm{diag}(Z) = \mathbf{1}_n$. Moreover, for all $i, j \in [n]$, we have $|Z_{ij}| = |\langle X_{i\bullet}, X_{j\bullet} \rangle| \leqslant \|X_{i\bullet}\|_2 \|X_{j\bullet}\|_2 \leqslant 1$. This proves that $Z \in \mathcal{C}'$ and so $\mathcal{C} \subset \mathcal{C}'$.

Let $Z' \in \mathrm{argmax}\left(\Re(\langle \mathbb{E}A, Z \rangle) : Z \in \mathcal{C}'\right\}$. Since $\mathcal{C}'$ is convex and the objective function $Z \to \Re(\langle \mathbb{E}A, Z \rangle)$ is linear (for real coefficients), Z' is one of the extreme points of $\mathcal{C}'$. Extreme points of $\mathcal{C}'$ are matrices $Z \in \mathbb{C}^{n \times n}$ such that $|Z_{ij}| = 1$ for all $i, j \in [n]$. We can then write each entry of Z' as $Z'_{ij} = e^{\iota\beta_{ij}}$ for some $0 \leqslant \beta_{ij} < 2\pi$ and now we obtain

$$\Re(\langle \mathbb{E}A, Z' \rangle) = \Re\left(\sum_{i,j=1}^{n} \mathbb{E}A_{ij}\overline{Z'_{ij}}\right) = \Re\left(\sum_{i \neq j}^{n} e^{-\sigma^2/2}e^{\iota\delta_{ij}}e^{-\iota\beta_{ij}}\right) + \Re\left(\sum_{i=1}^{n} e^{\iota\delta_{ii}}e^{-\iota\beta_{ii}}\right)$$

$$= \sum_{i \neq j} e^{\sigma^2/2}\cos(\delta_{ij} - \beta_{ij}) + \sum_{i}\cos(\delta_{ii} - \beta_{ii}) \leqslant e^{\sigma^2/2}(n^2 - n) + n.$$

The maximal value $e^{\sigma^2/2}(n^2 - n) + n$ is attained only for $\beta_{ij} = \delta_{ij}$ for all $i, j \in [n]$, that is for $Z' = (e^{\iota\delta_{ij}})_{i,j=1,\ldots,n} = Z^*$. But we have $Z^* \in \mathcal{C}$ and $\mathcal{C} \subset \mathcal{C}'$, so Z^* is the only maximizer of $Z \to \Re(\langle \mathbb{E}A, Z \rangle)$ on $\mathcal{C}$. But for all $Z \in \mathcal{C}$ we have $\langle \mathbb{E}A, Z \rangle = \overline{x^*}^\top Zx^* \in \mathbb{R}$, then Z^* is the only maximizer of $Z \to \langle \mathbb{E}A, Z \rangle$ over $\mathcal{C}$. ∎

6.1 Curvature of the objective function

Proposition 4 *For $\theta = e^{-\sigma^2/2}/2$, we have for all $Z \in \mathcal{C}$, $\langle \mathbb{E}A, Z^* - Z \rangle \geqslant \theta \|Z^* - Z\|_2^2$.*

Proof Let $Z = (z_{ij}e^{\iota\beta_{ij}})_{i,j=1}^{n} \in \mathcal{C}$ where $z_{ij} \in \mathbb{R}$ and $0 \leqslant \beta_{ij} < 2\pi$ for all $i, j \in [n]$. Since $Z_{ii}^* = Z_{ii} = 1$ for all $i \in [n]$, we have, on one side, $\langle \mathbb{E}A, Z^* - Z \rangle = e^{-\sigma^2/2}\overline{x^*}^\top(Z^* - Z)x^* \in \mathbb{R}$, and so

$$\langle \mathbb{E}A, Z^* - Z \rangle = \Re(\langle \mathbb{E}A, Z^* - Z \rangle) = \Re\left(\sum_{i,j=1}^{n} \mathbb{E}A_{ij}\overline{(Z^* - Z)_{ij}}\right)$$

$$= \Re\left(\sum_{i,j=1}^{n} e^{-\sigma^2/2}e^{\iota\delta_{ij}}\left(e^{-\iota\delta_{ij}} - z_{ij}e^{-\iota\beta_{ij}}\right)\right)$$

$$= e^{-\sigma^2/2}\Re\left(\sum_{i,j=1}^{n} 1 - z_{ij}e^{\iota(\delta_{ij} - \beta_{ij})}\right) = e^{-\sigma^2/2}\sum_{i,j=1}^{n}(1 - z_{ij}\cos(\delta_{ij} - \beta_{ij})). \tag{37}$$

On the other side, we proved in the proof of (18) that $\mathcal{C} \subset \{Z \in \mathbb{C}^{n \times n} : |Z_{ij}| \leqslant 1, \forall i, j \in [n]\}$. So we have $|z_{ij}| \leqslant 1$ for all $i, j \in [n]$ and

$$\|Z^* - Z\|_2^2 = \sum_{i,j=1}^{n} |(Z^* - Z)_{ij}|^2 = \sum_{i,j=1}^{n} |e^{\iota \delta_{ij}} - z_{ij} e^{\iota \beta_{ij}}|^2 = \sum_{i,j=1}^{n} |1 - z_{ij} e^{\iota(-\delta_{ij} + \beta_{ij})}|^2$$

$$= \sum_{i,j=1}^{n} (1 - z_{ij} \cos(\beta_{ij} - \delta_{ij}))^2 + z_{ij}^2 \sin^2(\beta_{ij} - \delta_{ij}) = \sum_{i,j=1}^{n} 1 - 2z_{ij} \cos(\beta_{ij} - \delta_{ij}) + z_{ij}^2$$

$$\leqslant 2 \sum_{i,j=1}^{n} (1 - z_{ij} \cos(\beta_{ij} - \delta_{ij})). \tag{38}$$

We conclude with (37) and (38). ∎

In fact, it follows from the proof of Proposition 4 that we have the following equality: for all $Z \in \mathcal{C}$,

$$\langle \mathbb{E}A, Z^* - Z \rangle = \theta \left(\|Z^* - Z\|_2^2 + \big\| |Z^*|^2 - |Z|^2 \big\|_1 \right)$$

where $|Z|^2 = (|Z_{ij}|^2)_{1 \leqslant i,j \leqslant n}$ (in particular, $|Z^*|^2 = (1)_{n \times n}$). We therefore know exactly how to characterize the curvature of the excess risk for the angular synchronization problem in terms of the ℓ_2 (to the square) and the ℓ_1 norms. Nevertheless, we will not use the extra term $\big\| |Z^*|^2 - |Z|^2 \big\|_1$ in the following.

6.2 Computation of the complexity fixed point $r_G^*(\Delta)$

It follows from the (global) curvature property of the excess risk for the angular synchronization problem obtained in Proposition 4 that for the curvature G function defined by $G(Z^* - Z) = \theta \|Z^* - Z\|_2^2, \forall Z \in \mathcal{C}$, we just have to compute the $r_G^*(\Delta)$ fixed point and then apply Corollary 1 in order to obtain statistical properties of $\hat{Z}$ (w.r.t. to both the excess risk and the G function). In this section, we compute the complexity fixed point $r_G^*(\Delta)$ for $0 < \Delta < 1$.

Following Proposition 4, the natural "local" subsets of $\mathcal{C}$ around Z^* which drive the statistical complexity of the synchronization problem are defined for all $r > 0$ by $\mathcal{C}_r = \{Z \in \mathcal{C} : \|Z - Z^*\|_2 \leqslant r\} = \mathcal{C} \cap (Z^* + r B_2^{n \times n})$.

Let $Z \in \mathcal{C}_r$. Denote by b_{ij}^R (resp. b_{ij}^I) the real (resp. imaginary) part of $b_{ij} = Z_{ij}^* \overline{Z_{ij} - Z_{ij}^*}$ for all $i, j \in [n]$. Since $\|Z - Z^*\|_2 \leqslant r$ we also have $\sum_{i,j}(b_{ij}^R)^2 + (b_{ij}^I)^2 \leqslant r^2$ and so

$$\langle A - \mathbb{E}A, Z - Z^* \rangle = \langle (S - \mathbb{E}S) \circ Z^*, Z - Z^* \rangle = 2 \Re \left(\sum_{i<j} (S_{ij} - \mathbb{E}S_{ij}) b_{ij} \right)$$

$$= 2 \sum_{i<j} (\cos(\sigma g_{ij}) - \mathbb{E}\cos(\sigma g_{ij})) b_{ij}^R - \sin(\sigma g_{ij}) b_{ij}^I$$

$$\leqslant 2r \sqrt{\sum_{i<j} (\cos(\sigma g_{ij}) - \mathbb{E}\cos(\sigma g_{ij}))^2 + (\sin(\sigma g_{ij}))^2}$$

$$\leqslant 2r \sqrt{1 - e^{-\sigma^2} + 2e^{-\sigma^2/2} \left(\sum_{i<j} \mathbb{E}\cos(\sigma g_{ij}) - \cos(\sigma g_{ij}) \right)}$$

where we used that $\mathbb{E}\cos(\sigma g) = \Re(\mathbb{E}e^{\iota g}) = e^{-\sigma^2/2}$ for $g \sim \mathcal{N}(0,1)$. Now its remains to get a high-probability upper bound on the sum of the centered cosinus of σg_{ij}. We use Bernstein's inequality (see Equation 41 below) to get such a bound. For all $t > 0$, with probability at least $1 - \exp(-t)$,

$$\frac{1}{\sqrt{N}} \sum_{i<j} \mathbb{E}\cos(\sigma g_{ij}) - \cos(\sigma g_{ij}) \leqslant \sqrt{2Vt} + \frac{2t}{3\sqrt{N}} \leqslant (1 - e^{-\sigma^2})\sqrt{t} + \frac{2t}{3\sqrt{N}}$$

for $N = n(n-1)/2$ and $V = \mathbb{E}\cos^2(\sigma g) - (\mathbb{E}\cos(\sigma g))^2 = (1/2)(1 - e^{-\sigma^2})^2$ (because $\mathbb{E}\cos^2(\sigma g) = (1/2)\mathbb{E}(1 + \cos(2\sigma g)) = (1/2)(1 + e^{-2\sigma^2})$ when $g \sim \mathcal{N}(0,1)$).

We now have all the ingredients to compute the fixed point $r_G^*(\Delta)$ for $0 < \Delta < 1$: for $\theta = e^{-\sigma^2/2}/2$ and $t = \log(1/\Delta)$,

$$r_G^*(\Delta) \leqslant \frac{4}{\theta}\left(1 - e^{-\sigma^2} + 2e^{-\sigma^2/2}\left((1 - e^{-\sigma^2})\sqrt{tN} + \frac{2t}{3}\right)\right) = \frac{32t}{3} + 8(1 - e^{-\sigma^2})(e^{\sigma^2/2} + 2\sqrt{tN}).$$

In particular, using $1 - e^{-\sigma^2} \leqslant \sigma^2$ and for $t = \epsilon\sigma^4 N$ (where $N = n(n-1)/2$) for some $0 < \epsilon < 1$, if $e^{\sigma^2/2} \leqslant 2\sigma^2\sqrt{\epsilon}N$ then $r_G^*(\Delta) \leqslant (128/3)\sigma^4 N\sqrt{\epsilon}$.

6.3 End of the proof of Theorem 4 and Corollary 2: application of Corollary 1

Take $\Delta = \exp(-\epsilon\sigma^4 N)$ (for $N = n(n-1)/2$), we have $r_G^*(\Delta) \leqslant (128/3)\sqrt{\epsilon}\sigma^4 N$ when $e^{\sigma^2/2} \leqslant 2\sqrt{\epsilon}\sigma^2 N$ (which holds for instance when $\sigma \leqslant \sqrt{\log(\epsilon N^2)}$) and so it follows from Corollary 1 (together with the curvature property in Section 6.1 and the computation of the fixed point $r_G^*(\Delta)$ from Section 6.2), that with probability at least $1 - \exp(-\epsilon\sigma^4 n(n-1)/2)$, $\theta\langle Z^* - Z\rangle_2^2 \leqslant \langle \mathbb{E}A, Z^* - Z\rangle \leqslant (128/3)\sqrt{\epsilon}\sigma^4 N$, which is the statement of Theorem 4.

Proof of Corollary 2: The oracle Z^* is the rank one matrix $x^*\overline{x^*}^\top$ which has n for largest eigenvalue and associated eigenspace $\{\lambda x^* : \lambda \in \mathbb{C}\}$. In particular, Z^* has a spectral gap $g = n$. Let $\hat{x} \in \mathbb{C}^n$ be a top eigenvector of $\hat{Z}$ with norm $\|\hat{x}\|_2 = \sqrt{n}$. It follows from Davis-Kahan Theorem (see, for example, Theorem 4.5.5 in Vershynin (2018) or Theorem 4 in Vu (2010)) that there exists an universal constant $c_0 > 0$ such that

$$\min_{z\in\mathbb{C}:|z|=1} \left\|\frac{\hat{x}}{\sqrt{n}} - z\frac{x^*}{\sqrt{n}}\right\|_2 \leqslant \frac{c_0}{g}\left\|\hat{Z} - Z^*\right\|_2$$

where $g = n$ is the spectral gap of Z^*. We conclude the proof of Corollary 2 using the upper bound on $\left\|\hat{Z} - Z^*\right\|_2$ from Theorem 4. ∎

7. Proofs of Theorems 7 and 8 (MAX-CUT)

In this section, we prove the two main results from Section 4.3 using our general methodology for Theorem 8 and the technique from Goemans and Williamson (1995) for Theorem 7.

7.1 Proof of Theorem 7

The proof of Theorem 7 follows the one from Goemans and Williamson (1995) up to a minor modification due to the fact that we use the SDP estimator $\hat{Z}$ instead of the oracle

Z^*. It is based on two tools. The first one is Grothendieck's identity: let $g \sim \mathcal{N}(0, \mathrm{I}_n)$ and $u, v \in \mathcal{S}_2^{n-1}$, we have:

$$\mathbb{E}[\mathrm{sign}(\langle g, u \rangle)\mathrm{sign}(\langle g, u \rangle)] = \frac{2}{\pi}\arcsin(\langle u, v \rangle) \tag{39}$$

and the identity: for all $t \in [-1, 1]$

$$1 - \frac{2}{\pi}\arcsin(t) = \frac{2}{\pi}\arccos(t) \geqslant 0.878(1 - t). \tag{40}$$

We now have enough tools to prove Theorem 7. The right-hand side inequality is trivial since $\mathrm{MAX} - \mathrm{CUT}(G) \leqslant \mathrm{SDP}(G)$. For the left-hand side, we denote by $\hat{X}_1, \ldots, \hat{X}_n$ (resp. $X_1^*, \ldots, X_n^*$) the n columns vectors in $\mathcal{S}_2^{n-1}$ of $\hat{Z}$ (resp. Z^*). We also consider the event Ω^* onto which

$$\langle \mathbb{E}B, Z^* - \hat{Z} \rangle \leqslant r^*(\Delta)$$

which hold with probability at least $1 - \Delta$ according to Theorem 1. On the event Ω^*, we have

$$\mathbb{E}\left[\mathrm{cut}(G, \hat{x})|\hat{Z}\right] = \mathbb{E}\left[\frac{1}{4}\sum_{i,j=1}^n A_{ij}^0(1 - \hat{x}_i\hat{x}_j)\right] = \frac{1}{4}\sum_{i,j=1}^n A_{ij}^0\left(1 - \mathbb{E}[\mathrm{sign}(\langle \hat{X}_i, g \rangle)\mathrm{sign}(\langle \hat{X}_j, g \rangle)]\right)$$

$$\overset{(i)}{=} \frac{1}{4}\sum_{i,j=1}^n A_{ij}^0\left(1 - \frac{2}{\pi}\arcsin(\langle \hat{X}_i, \hat{X}_j \rangle)\right) = \frac{1}{2\pi}\sum_{i,j=1}^n A_{ij}^0\arccos(\langle \hat{X}_i, \hat{X}_j \rangle)$$

$$\overset{(ii)}{\geqslant} \frac{0.878}{4}\sum_{i,j=1}^n A_{ij}^0(1 - \langle \hat{X}_i, \hat{X}_j \rangle)$$

$$= \frac{0.878}{4}\sum_{i,j=1}^n A_{ij}^0(1 - \langle X_i^*, X_j^* \rangle) + \frac{0.878}{4}\sum_{i,j=1}^n A_{ij}^0(\langle X_i^*, X_j^* \rangle - \langle \hat{X}_i, \hat{X}_j \rangle)$$

$$= \frac{0.878}{4}\langle A^0, J - Z^* \rangle + \frac{0.878}{4}\langle A^0, Z^* - \hat{Z} \rangle = 0.878\mathrm{SDP}(G) - \frac{0.878}{4}\langle \mathbb{E}B, Z^* - \hat{Z} \rangle$$

$$\geqslant 0.878\,\mathrm{SDP}(G) - \frac{0.878}{4}r^*(\Delta)$$

where we used (39) in *(i)* and (40) in *(ii)*.

7.2 Proof of Theorem 8

For the MAX-CUT problem, we do not use any localization argument; we therefore use the (likely sub-optimal) global approach. The methodology is very close to the one used in Guédon and Vershynin (2016) for the community detection problem. In particular, we use both Bernstein and Grothendieck inequalities to compute high-probability upper bound for $r^*(\Delta)$. We recall theses two tools now. First Bernstein's inequality: if $Y_1, \ldots, Y_N$ are N independent centered random variables such that $|Y_i| \leqslant M$ a.s. for all $i = 1, \ldots, N$ then for all $t > 0$, with probability at least $1 - \exp(-t)$,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N Y_i \leqslant \sigma\sqrt{2t} + \frac{2Mt}{3\sqrt{N}}, \tag{41}$$

where $\sigma^2 = (1/N)\sum_{i=1}^N \mathrm{var}(Y_i)$. The second tool is Grothendieck inequality Grothendieck (1956) (see also Pisier (2012) or Theorem 3.4 in Guédon and Vershynin (2016)): if $C \in \mathbb{R}^{n \times n}$ then

$$\sup_{Z \in \mathcal{C}} \langle C, Z \rangle \leqslant K_G \|C\|_{cut} = K_G \max_{s,t \in \{-1,1\}^n} \sum_{i,j=1}^n C_{ij} s_i t_j \tag{42}$$

where $\mathcal{C} = \{Z \succeq 0 : Z_{ii} = 1, i = 1, \ldots, n\}$ and K_G is an absolute constant, called the Grothendieck constant.

In order to apply Theorem 1, we just have to compute the fixed point $r^*(\Delta)$. As announced, we use the global approach and Grothendieck inequality (42) to get

$$\sup_{Z \in \mathcal{C}:\langle \mathbb{E}B, Z^* - Z \rangle \leqslant r} \langle B - \mathbb{E}B, Z - Z^* \rangle \leqslant \sup_{Z \in \mathcal{C}} \langle B - \mathbb{E}B, Z - Z^* \rangle \leqslant 2K_G \|B - \mathbb{E}B\|_{cut} \tag{43}$$

because $Z^* \in \mathcal{C}$. It follows from Bernstein's inequality (41) and a union bound that for all $t > 0$, with probability at least $1 - 4^n \exp(-t)$,

$$\|B - \mathbb{E}B\|_{cut} = \sup_{s,t \in \{\pm 1\}^n} \sum_{1 \leqslant i < j \leqslant n} (B_{ij} - \mathbb{E}B_{ij})(s_i t_j + s_j t_i) \leqslant 2\sqrt{\frac{(1-p)n(n-1)t}{p}} + \frac{4t}{3}.$$

Therefore, for $t = 2n \log 4$, with probability at least $1 - 4^{-n}$,

$$r^*(\Delta) \leqslant \|B - \mathbb{E}B\|_{cut} \leqslant 2n\sqrt{\frac{(2 \log 4)(1-p)(n-1)}{p}} + \frac{8n \log 4}{3}$$

for $\Delta = 4^{-n}$. Then the result follows from Theorem 1.

8. Numerical experiments

This section contains the outcome of numerical experiments on the three application problems considered: signed clustering, MAX-CUT, and angular synchronization.

8.1 Signed Clustering

To assess the effectiveness of the SDP relaxation, we consider the following experimental setup. We generate synthetic networks following the signed stochastic block model (SSBM) previously described in Section 4.1.1, with $K = 5$ communities. To quantify the effectiveness of the SDP relaxation, we compare the accuracy of a suite of algorithms from the signed clustering literature, *before* the SDP relaxation (i.e., when we perform these algorithms directly on A) and *after* the SDP relaxation (i.e., when we perform the very same algorithms on $\hat{Z}$). To measure the recovery quality of the clustering results, for a given indicator set $x_1, \ldots, x_K$, we rely on the error rate consider in Chiang et al. (2012), defined as

$$\gamma = \sum_{c=1}^K \frac{x_c^T A_{com}^- x_c + x_c^T L_{com}^+ x_c}{n^2}, \tag{44}$$

where x_c denotes a cluster indicator vector, A_{com} $(= \mathbb{E}A)$ is the complete K-weakly balanced ground truth network – with 1's on the diagonal blocks corresponding to inter-cluster edges,

and -1 otherwise – with $A_{com} = A_{com}^+ - A_{com}^-$, and L_{com}^+ denotes the combinatorial graph Laplacian corresponding to A_{com}^-. Note that $x_c^T A_{com}^- x_c$ counts the number of violations within the clusters (since negative edges should not be placed within clusters) and $x_c^T L_{com}^+ x_c$ counts the number of violations across clusters (since positive edges should not belong to the cut). Overall, (44) essentially counts the fraction of intra-cluster and inter-cluster edge violations, with respect to the full ground truth matrix. Note that this definition can also be easily adjusted to work on real data sets, where the ground truth matrix A_{com} is not available, which one can replace with the empirical observation A.

In terms of the signed clustering algorithms compared, we consider the following algorithms from the literature. One straightforward approach is to simply rely on the spectrum of the observed adjacency matrix A. Kunegis et al. (2010) proposed spectral tools for clustering, link prediction, and visualization of signed graphs, by solving a 2-way "signed" ratio-cut problem based on the combinatorial Signed Laplacian Hou (2005) $\bar{L} = \bar{D} - A$, where $\bar{D}$ is a diagonal matrix with $\bar{D}_{ii} = \sum_{i=1}^n |A_{ij}|$. The same authors proposed signed extensions for the case of the random-walk Laplacian $\bar{L}_{\mathrm{rw}} = I - \bar{D}^{-1}A$, and the symmetric graph Laplacian $\bar{L}_{\mathrm{sym}} = I - \bar{D}^{-1/2}A\bar{D}^{-1/2}$, the latter of which is particularly suitable for skewed degree distributions. Finally, the last algorithm we considered is BNC of Chiang et al. Chiang et al. (2012), who introduced a formulation based on the *Balanced Normalized Cut* objective

$$\min_{\{x_1,\ldots,x_K\}\in\mathcal{I}} \left(\sum_{c=1}^K \frac{x_c^T(D^+ - A)x_c}{x_c^T \bar{D} x_c} \right). \tag{45}$$

which, in light of the decomposition $D^+ - A = D^+ - (A^+ - A^-) = D^+ - A^+ + A^- = L^+ + A^-$, is effectively minimizing the number of violations in the clustering procedure.

In our experiments, we first compute the error rate γ_{before} of all algorithms on the original SSBM graph (shown in Column 1 of Figure 1), and then we repeat the procedure but with the input to all signed clustering algorithms being given by the output of the SDP relaxation, and denote the resulting recovery error by γ_{after}. The third column of the same Figure 1 shows the difference in errors $\gamma_\delta = \gamma_{before} - \gamma_{after}$ between the first and second columns, while the fourth column contains a histogram of the error differences γ_δ. This altogether illustrates the fact that the SDP relaxation does improve the performance of all signed clustering algorithms, except $\bar{L}$, and could effectively be used as a denoising pre-processing step. One potential reason why the SDP pre-processing step does not improve on the accuracy of $\bar{L}$ could stem from the fact that $\bar{L}$ has a good performance to begin with on examples where the clusters have equal sizes and the degree distribution is homogeneous. It would be interesting to further compare the results in settings with skewed degree distributions, such as the classical Barabási-Albert model Albert and Barabási (2002).

8.2 MAX-CUT

For the MAX-CUT problem, we consider two sets of numerical experiments. First, we consider a version of the stochastic block model which essentially perturbs a complete bipartite graph

$$\mathbf{B} = \begin{vmatrix} \mathbf{0}_{n_1 \times n_1} & \mathbf{1}_{n_1 \times n_2} \\ \mathbf{1}_{n_2 \times n_1} & \mathbf{0}_{n_2 \times n_2} \end{vmatrix}, \tag{46}$$
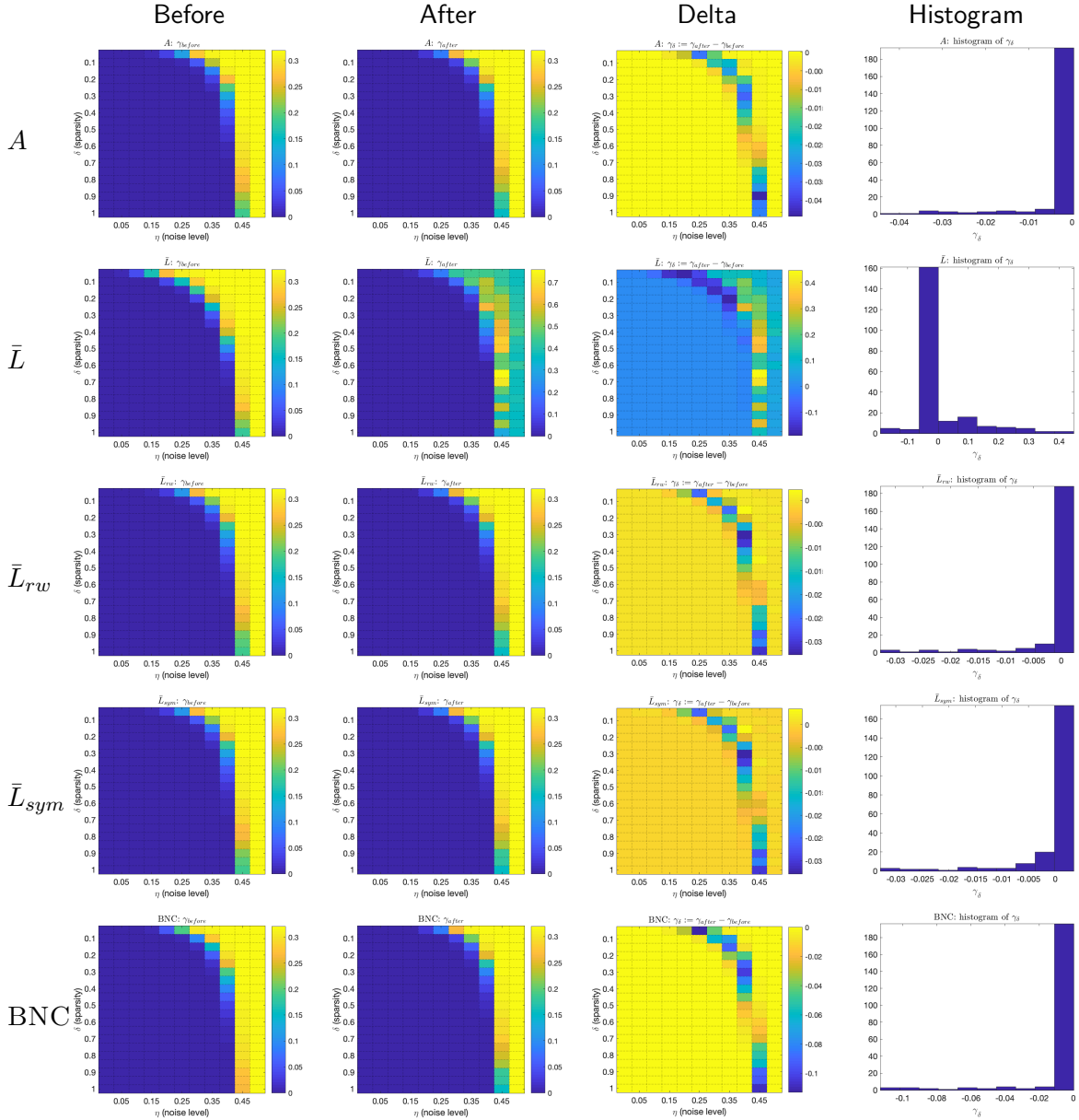
Figure 1: Summary of results for the Signed Clustering problem. The first column denotes the recovery error **before** the SDP relaxation step, meaning that we consider a number of signed clustering algorithms from the literature which we apply directly the initial adjacency matrix A. The second column contains the results when applying the same suite of algorithms **after** the SDP relaxation. The third column shows the difference in errors between the first and second columns, while the fourth column contains a histogram of the delta errors. This altogether illustrates the fact the SDP relaxation does improve the performance of all signed clustering algorithms except $\bar{L}$. Results are averaged over 20 runs.

where $\mathbf{1}_{n_1 \times n_2}$ (respectively, $\mathbf{0}_{n_1 \times n_2}$) denotes an $n_1 \times n_2$ matrix of all ones, respectively, all zeros. In our experiments, we set $n_1 = n_2 = \frac{n}{2}$, and fix $n = 500$. We perturb $\mathbf{B}$ by deleting edges across the two partitions, and inserting edges within each partition. More specifically, we generated the *full* adjacency matrix A^0 from $\mathbf{B}$ by adding edges independently with probability η within each partition (i.e., along the diagonal blocks in (46)). Finally, we denote by A the masked version we observe, $A = A^0 \circ S$, where S denotes the adjacency matrix of an Erdős-Rényi(n, δ) graph. The graph shown in Figure 2 is an instance of the above generative model. Note that for small values of η we expect the maximum cut to
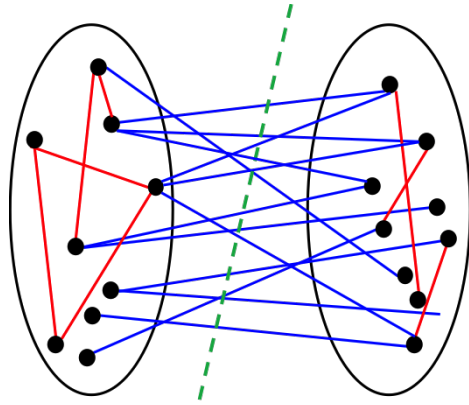

Figure 2: Illustration of MAX-CUT in the setting of a perturbation of a complete bipartite graph.

occur across the initial partition $\mathcal{P}_{\mathbf{B}}$ in the clean bipartite graph $\mathbf{B}$, which we aim to recover as we sparsify the observed graph A. The heatmap in the left of Figure 3 shows the Adjusted Rand Index (ARI) between the initial partition $\mathcal{P}_{\mathbf{B}}$ and the partition of the MAX-CUT SDP relaxation in (65), as we vary the noise parameter η and the sparsity δ. As expected, for a fix level of noise η, we are able to recover the hypothetically optimal MAX-CUT , for suitable levels of the sparsity parameter. The heatmap in the right of Figure 3 shows the computational running time, as we vary the two parameter, showing that the MANOPT solver takes the longest to solve dense noisy problems, as one would expect.

In the second set of experiments shown in Figure 4, we consider a graph A^0 chosen at random from the collection[3] of graphs known in the literature as the GSET, where we vary the sparsity level δ, and show the MAX-CUT value attained on the original full graph A^0, but using the MAX-CUT partition computed by the SDP relaxation (65) on the sparsified graph A.

8.3 Angular Synchronization

For the angular synchronization problem, we consider the following experimental setup, by assessing the quality of the recovered angular solution from the SDP relaxation, as we vary the two parameters of interest. In the x-axis in the plots from Figures 5 and 6 we vary the
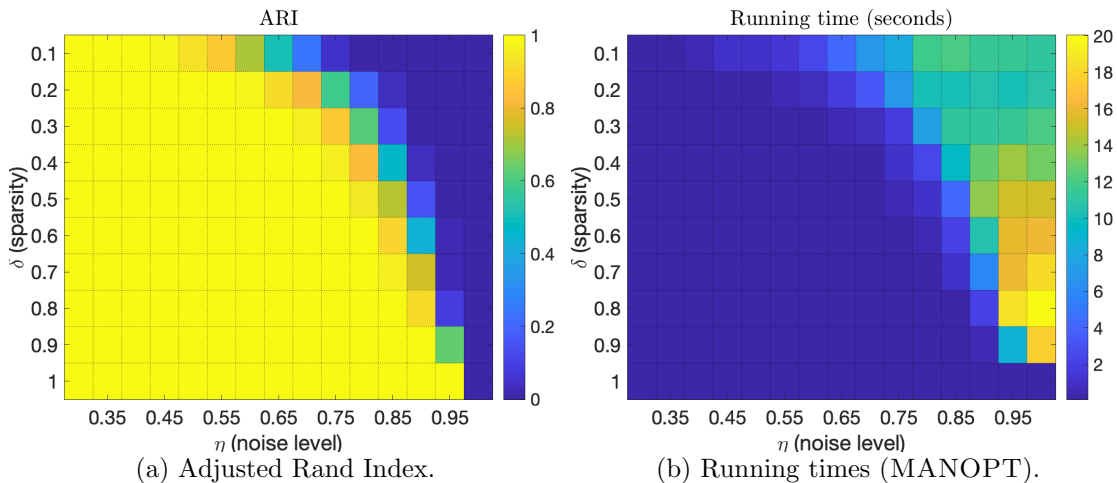
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3. http://web.stanford.edu/~yyye/yyye/Gset/

(a) Adjusted Rand Index. (b) Running times (MANOPT).

Figure 3: Numerical results for MAX-CUT on a perturbed complete bipartite graph, as we vary the noise level η and the sampling sparsity δ. Results are averaged over 20 runs.
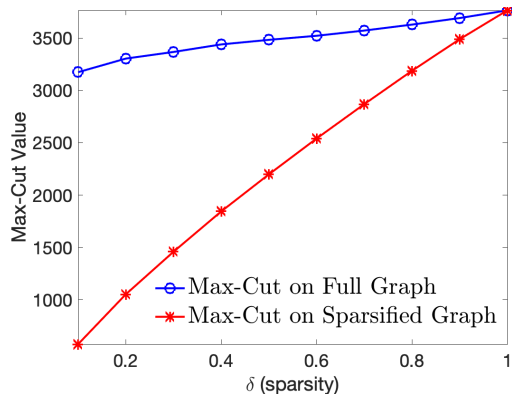

Figure 4: Max-Cut results for the G53 benchmark graph (from the GSET collection) with $n = 1000$ nodes and average degree ≈ 12. Results are averaged over 20 runs.

noise level σ, under two different noise models, Gaussian and outliers. On the y-axis, we vary the sparsity of the sampling graph.

We measure the quality of the recovered angles via the Mean Squared Error (MSE), defined as follows. Since a solution can only be recovered up to a global shift, one needs an MSE error that mods out such a degree of freedom. The following MSE is also more broadly applicable for the case when the underlying group is the orthogonal group $O(d)$, as opposed to $SO(2)$, as in the present work, where one can replace the unknown angles $\theta_1, \ldots, \theta_n$ with their respective representation as 2×2 rotation matrices $h_1, \ldots, h_n \in O(2)$. To that end, we look for an optimal orthogonal transformation $\hat{O} \in O(2)$ that minimizes the sum of squared distances between the estimated orthogonal transformations and the ground truth measurements

$$\hat{O} = \operatorname*{argmin}_{O \in O(2)} \sum_{i=1}^{n} \|h_i - O\hat{h}_i\|_F^2, \tag{47}$$

35

where $\hat{h}_1, \ldots, \hat{h}_n \in O(2)$ denote the 2×2 rotation matrix representation of the estimated angles $\hat{\theta}_1, \ldots, \hat{\theta}_n$. In other words, $\hat{O}$ is the optimal solution to the alignment problem between two sets of orthogonal transformations, in the least-squares sense. Following the analysis of Singer and Shkolnisky (2011), and making use of properties of the trace, one arrives at

$$
\begin{aligned}
\sum_{i=1}^n \|h_i - O\hat{h}_i\|_F^2 &= \sum_{i=1}^n \text{Trace}\left[\left(h_i - O\hat{h}_i\right)\left(h_i - O\hat{h}_i\right)^T\right] \\
&= \sum_{i=1}^n \text{Trace}\left[2I - 2O\hat{h}_i h_i^T\right] = 4n - 2\,\text{Trace}\left[O\sum_{i=1}^n \hat{h}_i h_i^T\right]. \quad (48)
\end{aligned}
$$

If we let Q denote the 2×2 matrix

$$
Q = \frac{1}{n}\sum_{i=1}^n \hat{h}_i h_i^T, \quad (49)
$$

it follows from (48) that the MSE is given by minimizing

$$
\frac{1}{n}\sum_{i=1}^n \|h_i - O\hat{h}_i\|_F^2 = 4 - 2Tr(OQ). \quad (50)
$$

In Arun et al. (1987) it is proven that $Tr(OQ) \leqslant Tr(VU^T Q)$, for all $O \in O(3)$, where $Q = U\Sigma V^T$ is the singular value decomposition of Q. Therefore, the MSE is minimized by the orthogonal matrix $\hat{O} = VU^T$ and is given by

$$
\text{MSE} \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^n \|h_i - \hat{O}\hat{h}_i\|_F^2 = 4 - 2\,\text{Trace}(VU^T U\Sigma V^T) = 4 - 2(\sigma_1 + \sigma_2), \quad (51)
$$

where σ_1, σ_2 are the singular values of Q. Therefore, whenever Q is an orthogonal matrix for which $\sigma_1 = \sigma_2 = 1$, the MSE vanishes. Indeed, the numerical experiments (on a log scale) in Figures 5 and 6 confirm that for noiseless data, the MSE is very close to zero. Furthermore, as one would expected, under favorable noise regimes and sparsity levels, we have almost perfect recovery, both by the SDP and the spectral relaxations, under both noise models.

9. Conclusions and future work

There are a number of other graph-based problems amenable to SDP relaxations, for which a similar theoretical analysis of their SDP-based estimators could be suitable. For example, the recent work of Cucuringu and Tyagi (2020) considered a problem motivated by geosciences and engineering applications of recovering a smooth unknown function $f : G \to \mathbb{R}$ (where $G = [a, b]$ is known) from noisy observations of its mod 1 values, which is also amenable to a solution based on an SDP relaxation solved via a Burer-Monteiro approach. Another potential application concerns the problem of clustering directed graphs, as in the very recent work of Cucuringu et al. (b) that proposed a spectral algorithm based on Hermitian matrices; this problem is also amenable to an SDP relaxation.

(a) Spectral relaxation. (b) SDP relaxation (solved via MANOPT).
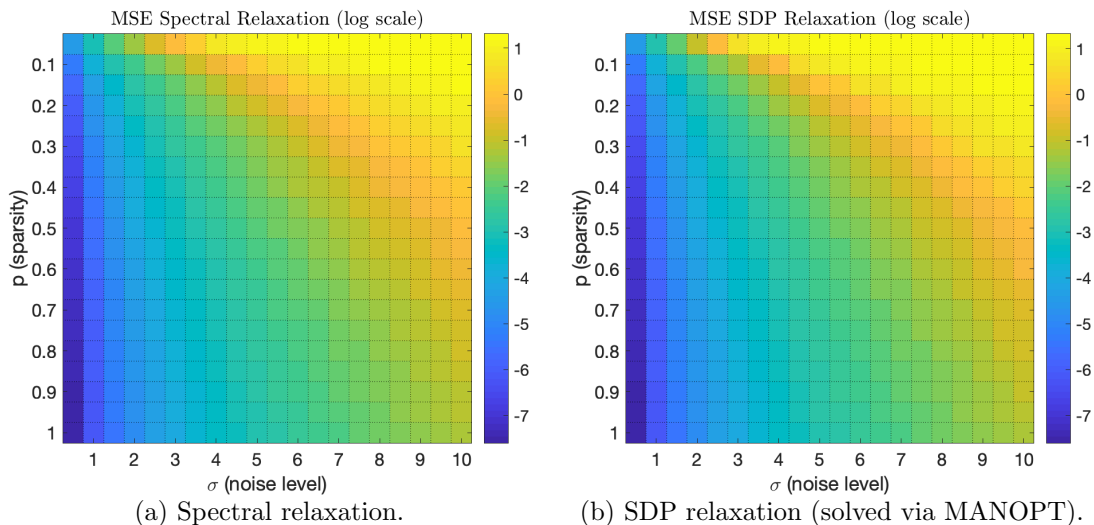
Figure 5: Recovery rates (MSE (51) - the lower the better) for angular synchronization with $n = 500$, under the Gaussian noise model, as we vary the noise level σ and the sparsity p of the measurement graph. Results are averaged over 20 runs.

(a) Spectral relaxation. (b) SDP relaxation (solved via MANOPT).
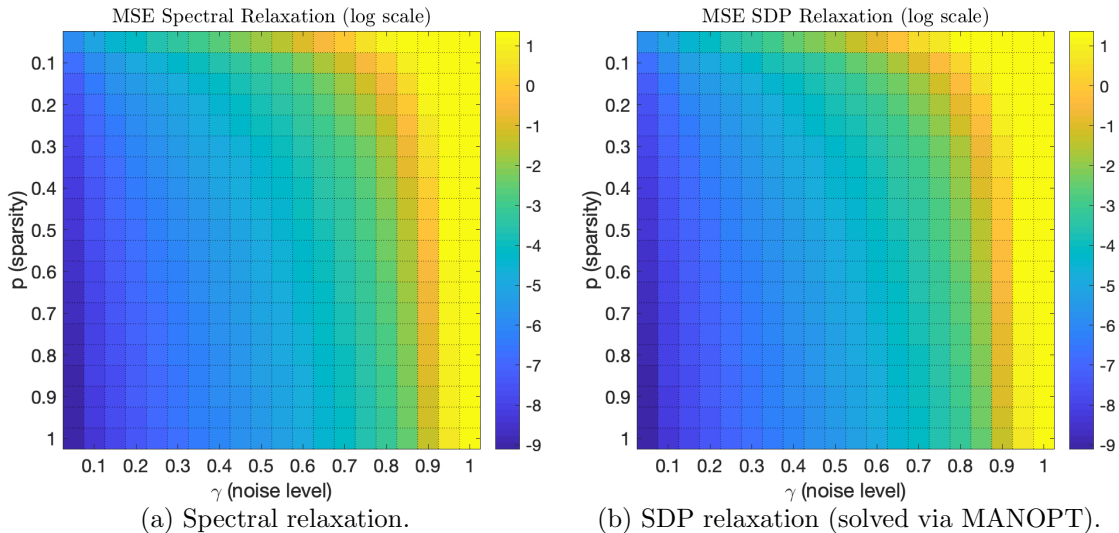
Figure 6: Recovery rates (MSE (51) - the lower the better) for angular synchronization with $n = 500$, under the Outlier noise model, as we vary the noise level γ and the sparsity p of the measurement graph. Results are averaged over 20 runs.

Our theoretical and practical findings show that running algorithms (such as spectral methods) directly on A may be improved by using first a SDP estimator such as $\hat{Z}$ and running the very same algorithms on $\hat{Z}$ (instead of A). Somehow, $\hat{Z}$ performs a pre-processing denoising step which improve the recovery of the hidden signal such as community vectors.

Acknowledgments

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Appendix A. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1

Denote $r^* = r^*(\Delta)$. Assume first that $r^* > 0$ (the case $r^* = 0$ is analyzed later). Let Ω^* be the event onto which for all $Z \in \mathcal{C}$ if $\langle \mathbb{E}A, Z^* - Z \rangle \leqslant r^*$ then $\langle A - \mathbb{E}A, Z - Z^* \rangle \leqslant (1/2)r^*$. By Definition of r^*, we have $\mathbb{P}[\Omega^*] \geqslant 1 - \Delta$.

Let $Z \in \mathcal{C}$ be such that $\langle \mathbb{E}A, Z^* - Z \rangle > r^*$ and define Z' such that $Z' - Z^* = \left(r^*/\langle \mathbb{E}A, Z^* - Z \rangle \right)(Z - Z^*)$. We have $\langle \mathbb{E}A, Z^* - Z' \rangle = r^*$ and $Z' \in \mathcal{C}$ because $\mathcal{C}$ is star-shaped in Z^*. Therefore, on the event Ω^*, $\langle A - \mathbb{E}A, Z' - Z^* \rangle \leqslant (1/2)r^*$ and so $\langle A - \mathbb{E}A, Z - Z^* \rangle \leqslant (1/2)\langle \mathbb{E}A, Z^* - Z \rangle$. It therefore follows that on the event Ω^*, if $Z \in \mathcal{C}$ is such that $\langle \mathbb{E}A, Z^* - Z \rangle > r^*$ then

$$\langle A, Z - Z^* \rangle \leqslant (-1/2)\langle \mathbb{E}A, Z^* - Z \rangle < -r^*/2$$

which implies that $\langle A, Z - Z^* \rangle < 0$ and therefore Z does not maximize $Z \to \langle A, Z \rangle$ over $\mathcal{C}$. As a consequence, we necessarily have $\langle \mathbb{E}A, Z^* - \hat{Z} \rangle \leqslant r^*$ on the event Ω^* (which holds with probability at least $1 - \Delta$).

Let us now assume that $r^* = 0$. There exists a decreasing sequence $(r_n)_n$ of positive real numbers tending to $r^* = 0$ such that for all $n \geqslant 0$, $\mathbb{P}[\Omega_n] \geqslant 1 - \Delta$ where Ω_n is the event $\Omega_n = \{\psi(r_n) \leqslant \theta/2\}$ where for all $r > 0$,

$$\psi(r) = \frac{1}{r} \sup_{Z \in \mathcal{C} : \langle \mathbb{E}A, Z^* - Z \rangle \leqslant r} \langle A - \mathbb{E}A, Z - Z^* \rangle.$$

Since $\mathcal{C}$ is star-shaped in Z^*, ψ is a non-increasing function and so $(\Omega_n)_n$ is a decreasing sequence (i.e. $\Omega_{n+1} \subset \Omega_n$ for all $n \geqslant 0$). It follows that $\mathbb{P}[\cap_n \Omega_n] = \lim_n \mathbb{P}[\Omega_n] \geqslant 1 - \Delta$. Let us now place ourselves on the event $\cap_n \Omega_n$. For all n, since Ω_n holds and $r_n > 0$, we can use the same argument as in first case to conclude that $\langle \mathbb{E}A, Z^* - \hat{Z} \rangle \leqslant r_n$. Since the latter inequality is true for all n (on the event $\cap_n \Omega_n$) and $(r_n)_n$ tends to zero, we conclude that $\langle \mathbb{E}A, Z^* - \hat{Z} \rangle \leqslant 0 = r^*$. ∎

Proof of Theorem 2

Let $r^* = r_G^*(\Delta)$. First assume that $r^* > 0$. Let $Z \in \mathcal{C}$ be such that $G(Z^* - Z) > r^*$. Let $f : \lambda \in [0, 1] \to G(\lambda(Z^* - Z))$. We have $f(0) = G(0) = 0$, $f(1) = G(Z^* - Z) > r^*$ and f is continuous. Therefore, there exists $\lambda_0 \in (0, 1)$ such that $f(\lambda_0) = r^*$. We let Z' be such that $Z' - Z^* = \lambda_0(Z - Z^*)$. Since $\mathcal{C}$ is star-shaped in Z^* and $\lambda_0 \in [0, 1]$ we have $Z' \in \mathcal{C}$. Moreover, $G(Z^* - Z') = r^*$. As a consequence, on the event Ω^* such that for all $Z \in \mathcal{C}$ if $G(Z^* - Z) \leqslant r^*$ then $\langle A - \mathbb{E}A, Z - Z^* \rangle \leqslant (1/2)r^*$, we have $\langle A - \mathbb{E}A, Z' - Z^* \rangle \leqslant (1/2)r^*$. The latter and Assumption 2 imply that, on Ω^*,

$$(1/2)r^* \geqslant \langle A, Z' - Z^* \rangle + \langle \mathbb{E}A, Z^* - Z' \rangle \geqslant \langle A, Z' - Z^* \rangle + G(Z^* - Z') \geqslant \langle A, Z' - Z^* \rangle + r^*$$

and so $\langle A, Z' - Z^* \rangle \leqslant -r^*/2$. Finally, using the definition of Z', we obtain

$$\langle A, Z - Z^* \rangle = (1/\lambda_0)\langle A, Z' - Z^* \rangle \leqslant -r^*/(2\lambda_0) < 0.$$

In particular, Z cannot be a maximizer of $Z \to \langle A, Z \rangle$ over $\mathcal{C}$ and so necessarily, on the event Ω^*, $G(Z^* - \hat{Z}) \leqslant r^*$.

Let us now consider the case where $r^* = 0$. Using the same approach as in the proof of Theorem 1, we only have to check that the function

$$\psi : r > 0 \to \frac{1}{r} \sup_{Z \in \mathcal{C} : G(Z^* - Z) \leqslant r} \langle A - \mathbb{E}A, Z - Z^* \rangle$$

is non-increasing. Let $0 < r_1 < r_2$. W.l.o.g. we may assume that there exists some $Z_2 \in \mathcal{C}$ such that $G(Z^* - Z_2) \leqslant r_2$ and $\psi(r_2) = \langle A - \mathbb{E}A, Z_2 - Z^* \rangle / r_2$. If $G(Z^* - Z_2) \leqslant r_1$ then $\psi(r_2) \leqslant (r_1/r_2)\psi(r_1) \leqslant \psi(r_1)$. If $G(Z^* - Z_2) > r_1$ then there exists $\lambda_0 \in (0,1)$ such that for $Z_1 = Z^* + \lambda_0(Z_2 - Z^*)$ we have $G(Z^* - Z_1) = r_1$ and $Z_1 \in \mathcal{C}$. Moreover, $r_1 = G(\lambda_0(Z^* - Z_2)) \leqslant \lambda_0 G(Z^* - Z_2) \leqslant \lambda_0 r_2$ and so $\lambda_0 \geqslant r_1/r_2$. It follows that

$$\psi(r_2) = \frac{1}{r_2}\langle A - \mathbb{E}A, Z_2 - Z^* \rangle = \frac{1}{\lambda_0 r_2}\langle A - \mathbb{E}A, Z_1 - Z^* \rangle \leqslant \frac{r_1}{\lambda_0 r_2}\psi(r_1) \leqslant \psi(r_1)$$

where we used that $\psi(r) > 0$ for all $r > 0$ because $Z^* \in \{Z \in \mathcal{C} : G(Z^* - Z) \leqslant r\}$ for all $r > 0$. $\blacksquare$

Appendix B. Additional proofs for Signed Clustering

Proof of Equation (10)

We recall that the cluster matrix $\bar{Z} \in \{0,1\}^{n \times n}$ is defined by $Z_{ij} = 1$ if $i \sim j$ and $Z_{ij} = 0$ when $i \not\sim j$ and $\alpha = \delta(p + q - 1)$. For all matrix $Z \in [0,1]^{n \times n}$, we have

$$\langle Z, \mathbb{E}A - \alpha J \rangle = \sum_{i,j=1}^{n} Z_{ij}(\mathbb{E}A_{ij} - \alpha) = \sum_{(i,j) \in \mathcal{C}^+} Z_{ij}(\mathbb{E}A_{ij} - \alpha) + \sum_{(i,j) \in \mathcal{C}^-} Z_{ij}(\mathbb{E}A_{ij} - \alpha)$$

$$= [\delta(2p-1) - \alpha] \sum_{(i,j) \in \mathcal{C}^+ : i \neq j} Y_{ij} + [\delta(2q-1) - \alpha] \sum_{(i,j) \in \mathcal{C}^-} Z_{ij} + (1 - \alpha)\sum_{i=1}^{n} Z_{ii}$$

$$= \delta(p - q)\left[\sum_{(i,j) \in \mathcal{C}^+} Y_{ij} - \sum_{(i,j) \in \mathcal{C}^-} Y_{ij}\right] + (1 - \alpha)\sum_{i=1}^{n} Z_{ii}.$$

The latter quantity is maximal for $Z \in [0,1]^{n \times n}$ such that $Z_{ij} = 1$ for $(i,j) \in \mathcal{C}^+$ and $Z_{ij} = 0$ for $(i,j) \in \mathcal{C}^-$, that is when $Z = \bar{Z}$. As a consequence $\{\bar{Z}\} = \text{argmax}_{Z \in [0,1]^{n \times n}} \langle Z, \mathbb{E}A - \alpha J \rangle$. Moreover, $\bar{Z} \in \mathcal{C} \subset [0,1]^{n \times n}$ so we also have that $\bar{Z}$ is the only solution to the problem $\max_{Z \in \mathcal{C}} \langle Z, \mathbb{E}A - \alpha J \rangle$ and so $\bar{Z} = Z^*$. $\blacksquare$

Proof of Proposition 2: control of $S_1(Z)$ adapted from Fei and Chen (2019)

The noise matrix W is symmetric and has been decomposed as $W = \Psi + \Psi^\top$ where Ψ has been defined in (62). For all $Z \in \mathcal{C} \cap (Z^* + rB_1^{n \times n})$, we have

$$
\begin{aligned}
S_1(Z) &= \langle \mathcal{P}(Z - Z^*), W \rangle = \langle \mathcal{P}(W), Z - Z^* \rangle \\
&= \langle UU^\top W, Z - Z^* \rangle + \langle WUU^\top, Z - Z^* \rangle - \langle UU^\top WUU^\top, Z - Z^* \rangle \\
&= 2\langle UU^\top W, Z - Z^* \rangle - \langle UU^\top WUU^\top, Z - Z^* \rangle \\
&= 2\langle UU^\top \Psi, Z - Z^* \rangle + 2\langle UU^\top \Psi^\top, Z - Z^* \rangle - \langle UU^\top (\Psi + \Psi^\top) UU^\top, Z - Z^* \rangle \\
&= 2\langle UU^\top \Psi, Z - Z^* \rangle + 2\langle UU^\top \Psi^\top, Z - Z^* \rangle - 2\langle UU^\top \Psi UU^\top, Z - Z^* \rangle \\
&= 2\langle UU^\top \Psi, Z - Z^* \rangle + 2\langle UU^\top \Psi^\top, Z - Z^* \rangle - 2\langle UU^\top \Psi, (Z - Z^*) UU^\top \rangle. \quad (52)
\end{aligned}
$$

An upper bound on $S_1(Z)$ follows from an upper bound on the three terms in the right side of (52). Let us show how to bound the first term. Similar arguments can be used to control the other two terms.

Let $V := UU^T \Psi$. Let us find a high-probability upper bound on the term $\langle UU^\top \Psi, Z - Z^* \rangle = \langle V, Z - Z^* \rangle$ uniformly over $Z \in \mathcal{C} \cap (Z^* + rB_1^{n \times n})$. For all $k \in [K]$, $i \in \mathcal{C}_k$ and $j \in [n]$, we have

$$
V_{ij} = \sum_{t=1}^n (UU^T)_{it} \Psi_{tj} = \sum_{t \in \mathcal{C}_k} \frac{1}{l_k} \Psi_{tj} = \frac{1}{l_k} \sum_{t \in \mathcal{C}_k} \Psi_{tj} = \frac{1}{l_k} \sum_{t \in \mathcal{C}_k} \Psi_{tj}.
$$

Therefore, given $j \in [n]$ the V_{ij}'s are all equal for $i \in \mathcal{C}_k$. We can therefore fix some arbitrary index $i_k \in \mathcal{C}_k$ and have $V_{ij} = V_{i_k j}$ for all $i \in \mathcal{C}_k$. Moreover, $(V_{i_k j} : k \in [K], j \in [n])$ is a family of independent random variables. We now have

$$
\begin{aligned}
\langle V, Z - Z^* \rangle &= \sum_{k \in [K]} \sum_{i \in \mathcal{C}_k} \sum_{j \in [n]} V_{ij}(Z - Z^*)_{ij} = \sum_{k \in [K]} \sum_{j \in [n]} l_k V_{i_k j} \sum_{i \in \mathcal{C}_k} \frac{(Z - Z^*)_{ij}}{l_k} \\
&= \sum_{k \in [K]} \sum_{j \in [n]} l_k V_{i_k j} w_{kj}
\end{aligned}
$$

which is a weighted sum of nK independent centered random variables $X_{k,j} := l_k V_{i_k j}$ with weights $w_{k,j} = (1/l_k) \sum_{i \in \mathcal{C}_k} (Z - Z^*)_{ij}$ for $k \in [K], j \in [n]$. We now idenfity some properties on the weights w_{kj}.

The weights are such that

$$
\sum_{k \in [K]} \sum_{j \in [n]} |w_{k,j}| \leqslant \frac{c_1 K}{n} \sum_{k \in [K]} \sum_{j \in [n]} \sum_{i \in \mathcal{C}_k} |(Z - Z^*)_{ij}| = \|Z - Z^*\|_1 \frac{c_1 K}{n} \leqslant \frac{c_1 r K}{n}
$$

which is equivalent to say that the weights vector $w = (w_{kj} : k \in [K], j \in [n])$ is in the ℓ_1^{Kn}-ball $(c_1 r K/n) B_1^{Kn}$. It is also in the unit ℓ_∞^{Kn}-ball since for all $k \in [K]$ and $j \in [n]$,

$$
|w_{k,j}| \leqslant \sum_{i \in \mathcal{C}_k} \frac{|(Z - Z^*)_{ij}|}{l_k} \leqslant \|Z - Z^*\|_\infty \leqslant 1.
$$

We therefore have $w \in B_\infty^{Kn} \cap (c_1 r K/n) B_1^{Kn}$ and so

$$
\sup_{Z \in \mathcal{C} \cap (Z^* + rB_1^{n \times n})} \langle V, Z - Z^* \rangle \leqslant \sup_{w \in B_\infty^{Kn} \cap (c_1 r K/n) B_1^{Kn}} \sum_{k \in [K], j \in [n]} X_{k,j} w_{k,j}. \quad (53)
$$

It remains to find a high-probability upper bound on the latter term. We can use the following lemma to that end.

Lemma 1 *Let $X_{k,j} = \sum_{t \in \mathcal{C}_k} \Psi_{tj}$ for $(k,j) \in [K] \times [n]$. For all $0 \leqslant R \leqslant Kn$, if $\lceil R \rceil \geqslant 2eKn \exp(-(9/32)n\rho/K)$ then with probability at least $1 - (\lceil R \rceil/(2eKn))^{\lceil R \rceil}$,*

$$\sup_{w \in B_\infty^{Kn} \cap RB_1^{Kn}} \sum_{(k,j) \in [K] \times [n]} X_{k,j} w_{k,j} \leqslant 4\sqrt{8c_0} R \sqrt{\frac{n\rho}{K} \log\left(\frac{2eKn}{\lceil R \rceil}\right)}.$$

Proof of Lemma 1. Let $N = Kn$ and assume that $1 \leqslant R \leqslant N$. We denote by $X_1^* \geqslant X_2^* \geqslant \cdots, \geqslant X_N^*$ (resp. $w_1^* \geqslant \cdots \geqslant w_N^*$) the non-decreasing rearrangement of $|X_{k,j}|$ (resp. $|w_{k,j}|$) for $(k,j) \in [K] \times [n]$. We have

$$\sup_{w \in B_\infty^N \cap RB_1^N} \sum_{(k,j) \in [K] \times [n]} X_{k,j} w_{k,j} \leqslant \sup_{w \in B_\infty^N \cap RB_1^N} \sum_{i=1}^N X_i^* w_i^*$$

$$\leqslant \sup_{w \in B_\infty^N} \sum_{i=1}^{\lceil R \rceil} X_i^* w_i^* + \sup_{w \in RB_1^N} \sum_{i=\lceil R \rceil+1}^N X_i^* w_i^* \leqslant \sum_{i=1}^{\lceil R \rceil} X_i^* + R X_{\lceil R \rceil+1}^* \leqslant 2 \sum_{i=1}^{\lceil R \rceil} X_i^*.$$

Moreover, for all $\tau > 0$, using a union bound, we have

$$\mathbb{P}\left(\sum_{i=1}^{\lceil R \rceil} X_i^* > \tau\right) = \mathbb{P}\left(\exists I \subset [K] \times [n] : |I| = \lceil R \rceil \text{ and } \sum_{(k,j) \in I} |X_{k,j}| > \tau\right)$$

$$= \mathbb{P}\left(\max_{I \subset [K] \times [n] : |I| = \lceil R \rceil} \max_{u_{k,j} = \pm 1, (k,j) \in I} \sum_{(k,j) \in I} u_{k,j} X_{k,j} > \tau\right)$$

$$\leqslant \sum_{I \subset [K] \times [n] : |I| = \lceil R \rceil} \sum_{u \in \{\pm 1\}^{\lceil R \rceil}} \mathbb{P}\left(\sum_{(k,j) \in I} X_{k,j} u_{k,j} > \tau\right)$$

$$= \sum_{I \subset [K] \times [n] : |I| = \lceil R \rceil} \sum_{u \in \{\pm 1\}^{\lceil R \rceil}} \mathbb{P}\left(\sum_{(k,j) \in I} \sum_{t \in \mathcal{C}_k} \Psi_{t,j} u_{k,j} > \tau\right).$$

Let us now control each term of the latter sum thanks to Bernstein inequality. The random variables $(\Psi_{t,j} : t, j \in [n])$ are independent with variances at most $\rho = \delta \max(1 - \delta(2p - 1)^2, 1 - \delta(2q-1)^2)$ since $\text{Var}(\Psi_{ij}) = 0$ when $i > j$ and $\text{Var}(\Psi_{ij}) = \text{Var}(A_{ij} - \mathbb{E}[A_{ij}]) = \text{Var}(A_{ij}) \leqslant \rho$ for $j \geqslant i$. Moreover, $|\Psi_{ij}| = 0$ when $j < i$ and $|\Psi_{ij}| = |W_{ij}| = |A_{ij} - \mathbb{E}A_{ij}| \leqslant 2$ for $j \geqslant i$ because $A_{ij} \in \{-1, 0, 1\}$. It follows from Bernstein's inequality that for all $I \subset [K] \times [n]$ satisfying $|I| = \lceil R \rceil$ and $u \in \{\pm 1\}^{\lceil R \rceil}$ that

$$\mathbb{P}\left(\sum_{(k,j) \in I} \sum_{t \in \mathcal{C}_k} \Psi_{t,j} u_{k,j} > \tau\right) \leqslant \exp\left(\frac{-\tau^2}{2\lceil R \rceil l_k \rho + 4\tau/3}\right) \leqslant \exp\left(\frac{-\tau^2}{2\lceil R \rceil c_0 n\rho/K + 4\tau/3}\right)$$

$$\leqslant \exp\left(\frac{-\tau^2}{4\lceil R \rceil c_0 n\rho/K}\right)$$

when $\tau \leqslant (3/2)\lceil R \rceil c_0 n\rho/K$. Therefore, $\sup_{w \in B_\infty^N \cap RB_1^N} \sum X_{k,j}w_{k,j} \leqslant 2\tau$ with probability at least

$$1 - \binom{N}{\lceil R \rceil} 2^{\lceil R \rceil} \exp\left(\frac{-\tau^2}{4\lceil R \rceil c_0 n\rho/K} \right) \geqslant 1 - \exp\left(\frac{-\tau^2}{8\lceil R \rceil c_0 n\rho/K} \right)$$

when

$$(3/2)\lceil R \rceil c_0 n\rho/K \geqslant \tau \geqslant \sqrt{8c_0}\lceil R \rceil \sqrt{\frac{n\rho}{K} \log\left(\frac{2eN}{\lceil R \rceil} \right)}$$

which is a non vacuous condition since $\lceil R \rceil \geqslant 2eN \exp(-(9/32)n\rho/K)$. The result follows, in the case $1 \leqslant R \leqslant N$, by taking $\tau = \sqrt{8c_0}\lceil R \rceil \sqrt{(n\rho/K) \log(2eN/\lceil R \rceil)}$ and using that $2R \geqslant \lceil R \rceil$ when $R \geqslant 1$.

For $0 \leqslant R \leqslant 1$, we have

$$\sup_{w \in B_\infty^{Kn} \cap RB_1^{Kn}} \sum_{(k,j) \in [K] \times [n]} X_{k,j}w_{k,j} = R \max_{(k,j) \in [K] \times [n]} |X_{k,j}|$$

and using Bernstein inequality as above we get that with probability at least $1 - \exp(-K\tau^2/(8c_0 n\rho))$, $\max_{(k,j) \in [K] \times [n]} |X_{k,j}| \leqslant \tau$ when $3c_0 n\rho/(2K) \geqslant \tau \geqslant \sqrt{8c_0 n\rho \log(nK)/K}$ which is a non vacuous condition when $9c_0 n\rho \geqslant 4K \log(nK)$. By taking $\tau = \sqrt{8c_0 n\rho \log(nK)/K}$, we obtain, that for all $0 \leqslant R \leqslant 1$, if $9c_0 n\rho \geqslant 4K \log(nK)$ then with probability at least $1 - 1/(nK)$,

$$\sup_{w \in B_\infty^{Kn} \cap RB_1^{Kn}} \sum_{(k,j) \in [K] \times [n]} X_{k,j}w_{k,j} \leqslant R\sqrt{\frac{8c_0 n\rho \log(nK)}{K}}.$$

∎

We apply Lemma 1 for $R = c_1 rK/n$ to control (53):

$$\mathbb{P}\left[\sup_{Z \in \mathcal{C} \cap (Z^* + rB_1^{n \times n})} \langle V, Z - Z^* \rangle \leqslant c_2 r\sqrt{\frac{K\rho}{n} \log\left(\frac{2eKn}{\lceil \frac{c_1 rK}{n} \rceil} \right)} \right] \geqslant 1 - \left(\frac{\lceil \frac{c_1 rK}{n} \rceil}{2eKn} \right)^{\lceil \frac{c_1 rK}{n} \rceil}$$

when $\lceil c_1 rK/n \rceil \geqslant 2eKn \exp(-(9/32)n\rho/K)$.

Using the same methodology, we can prove exactly the same result for the quantity

$$\sup_{Z \in \mathcal{C} \cap (Z^* + rB_1^{n \times n})} \langle UU^\top \Psi^\top, Z - Z^* \rangle.$$

We can also use the same method to upper bound $\sup_{Z \in \mathcal{C} \cap (Z^* + rB_1^{n \times n})} \langle UU^\top \Psi^\top, (Z - Z^*)UU^\top \rangle$, we simply have to check that the weights vector $w' = (w'_{kj} : k \in [K], j \in [n])$ where $w'_{kj} = (1/l_k) \sum_{i \in \mathcal{C}_k} [(Z - Z^*)UU^\top]_{ij}$ is also in $B_\infty^{Kn} \cap (c_1 rK/n)B_1^{Kn}$ for any $Z \in \mathcal{C}$ such that $\|Z - Z^*\|_1 \leqslant r$. This is indeed the case, since we have for all $i \in [n]$, $k' \in [K]$ and $j \in \mathcal{C}_{k'}$, $[(Z - Z^*)UU^\top]_{ij} = \sum_{p=1}^n (Z - Z^*)_{ip}(UU^\top)_{pj} = \sum_{p \in \mathcal{C}_{k'}} (Z - Z^*)_{ip}/l_{k'}$ which is therefore constant for all elements in $j \in \mathcal{C}_{k'}$. Therefore, we have

$$\sum_{k \in [K]} \sum_{j \in [n]} |w'_{kj}| = \sum_{k \in [K]} \sum_{k' \in [K]} \sum_{j \in \mathcal{C}_{k'}} \left| \frac{1}{l_k l_{k'}} \sum_{i \in \mathcal{C}_k} \sum_{p \in \mathcal{C}_{k'}} (Z - Z^*)_{ip} \right|$$

$$\leqslant \sum_{k \in [K]} \sum_{k' \in [K]} \sum_{j \in \mathcal{C}_{k'}} \frac{1}{l_k l_{k'}} \sum_{i \in \mathcal{C}_k} \sum_{p \in \mathcal{C}_{k'}} |(Z - Z^*)_{ip}| \leqslant \|Z - Z^*\|_1 \frac{c_1 K}{n} \leqslant \frac{c_1 rK}{n}$$

and for all $k' \in [K]$ and $j \in \mathcal{C}_{k'}$,

$$|w'_{kj}| = \left| \frac{1}{l_k l_{k'}} \sum_{i \in \mathcal{C}_k} \sum_{p \in \mathcal{C}_{k'}} (Z - Z^*)_{ip} \right| \leqslant \|Z - Z^*\|_\infty \leqslant 1.$$

Therefore, $w' \in B_\infty^{Kn} \cap (c_1 rK/n) B_1^{Kn}$ and we obtain exactly the same upper bound for the three terms in (52). This concludes the proof of Proposition 2.

Proof of Proposition 3: control of the $S_2(Z)$ term from Fei and Chen (2019)

In this section, we prove Proposition 3. We follow the proof from Fei and Chen (2019) but we only consider the "dense case" which is when $n\delta\nu \geqslant \log n$ – we recall that $\nu = \max(2p - 1, 1 - 2q)$. For a similar uniform control of $S_2(Z)$ in the "sparse case ", when $c_0 \leqslant n\delta\nu \leqslant \log n$ for some absolute constant c_0, we refer the reader to Fei and Chen (2019).

For all $Z \in \mathcal{C}$, we have $S_2(Z) = \langle \mathcal{P}^\perp(Z - Z^*), W \rangle = \langle \mathcal{P}^\perp(Z), W \rangle$ because, by construction of the projection operator, $\mathcal{P}^\perp(Z^*) = 0$. Therefore, $S_2(Z) \leqslant \|\mathcal{P}^\perp(Z)\|_* \|W\|_{\text{op}}$ where $\|\cdot\|_*$ denotes the nuclear norm (i.e. the sum of singular values) and $\|\cdot\|_{\text{op}}$ denotes the operator norm (i.e. maximum of the singular value). In the following Lemma 2, we prove an upper bound for $\|\mathcal{P}^\perp(Z)\|_*$ and then, we will obtain a high-probability upper bound onto $\|W\|_{\text{op}}$.

Lemma 2 *For all $Z \in \mathcal{C} \cap (Z^* + rB_1^{n \times n})$, we have*

$$\left\| \mathcal{P}^\perp(Z) \right\|_* = \text{Tr}(\mathcal{P}^\perp(Z)) \leqslant \frac{c_1 k}{n} \|Z - Z^*\|_1 \leqslant \frac{c_1 Kr}{n}.$$

Proof Since $Z \succeq 0$ so it is for $(\mathbb{I}_n - UU^\top)Z(\mathbb{I}_n - UU^\top)$ and so $\mathcal{P}^\perp(Z) = (\mathbb{I}_n - UU^\top)Z(\mathbb{I}_n - UU^\top) \succeq 0$ therefore $\left\| \mathcal{P}^\perp(Z) \right\|_* = \text{Tr}(\mathcal{P}^\perp(Z))$. Next, we bound the trace of $\mathcal{P}^\perp(Z)$.

Since $\mathbb{I}_n - UU^\top$ is a projection operator, it is symmetric and $(\mathbb{I}_n - UU^\top)^2 = \mathbb{I}_n - UU^\top$, moreover, $\text{Tr}(Z) = n = \text{Tr}(Z^*)$ when $Z \in \mathcal{C}$ so

$$\begin{aligned}
\text{Tr}(\mathcal{P}^\perp(Z)) &= \text{Tr}(\mathcal{P}^\perp(Z - Z^*)) = \text{Tr}((\mathbb{I}_n - UU^\top)(Z - Z^*)(\mathbb{I}_n - UU^\top)) \\
&= \text{Tr}((\mathbb{I}_n - UU^T)^2(Z - Z^*)) = \text{Tr}((\mathbb{I}_n - UU^T)(Z - Z^*)) \\
&= \text{Tr}(Z) - \text{Tr}(Z^*) + \text{Tr}(UU^T(Z^* - Z)) = \sum_{i,j}(UU^T)_{ij}(Z^* - Z)_{ij} \\
&= \sum_{k \in [K]} \sum_{i,j \in \mathcal{C}_k} \frac{1}{l_k}(Z^* - Z)_{ij} \overset{(i)}{=} \sum_{k \in [K]} \frac{1}{l_k} \sum_{i,j \in \mathcal{C}_k} |(Z^* - Z)_{ij}| \\
&\leqslant \frac{c_1 K}{n} \sum_{k \in [K]} \sum_{i,j \in \mathcal{C}_k} |(Z^* - Z)_{ij}| \leqslant \frac{c_1 K}{n} \|Z - Z^*\|_1
\end{aligned}$$

where we used in *(i)* that for i and j in a same community, we have $Z^*_{ij} = 1$ and $Z_{ij} \in [0, 1]$, thus $(Z^* - Z)_{ij} = |(Z^* - Z)_{ij}|$. Finally, when Z is in the localized set $\mathcal{C} \cap (Z^* + rB_1^{n \times n})$, we have $\|Z - Z^*\|_1 \leqslant r$ which concludes the proof. ∎

Now, we obtain a high-probability upper bound on $\|W\|_{\mathrm{op}}$. In the following, we apply this result in the "dense case" (i.e. $n\delta\nu \geqslant \log n$) to get the uniform bound onto $S_2(Z)$ over $Z \in \mathcal{C} \cap (Z^* + rB_1^{n\times n})$.

Lemma 3 (Lemma 4 in Fei and Chen (2019)) *With probability at least* $1-\exp(-\delta\nu n)$, $\|W\|_{op} \leqslant 16\sqrt{\delta\nu n} + 168\sqrt{\log(n)}$.

Proof Let A' be an independent copy of A and $R \in \mathbb{R}^{n\times n}$ be a random symmetric matrix independent from both A and A' whose sub-diagonal entries are independent Rademacher random variables. Using a symmetrization argument (see Chapter 2 in Koltchinskii (2011) or Chapter 2.3 in van der Vaart and Wellner (1996)), we obtain for $W = A - \mathbb{E}A$,

$$\mathbb{E}\|W\|_{\mathrm{op}} = \mathbb{E}\|A - \mathbb{E}A'\|_{\mathrm{op}} \overset{(i)}{\leqslant} \mathbb{E}\|A - A'\|_{\mathrm{op}} \overset{(ii)}{=} \mathbb{E}\|(A - A') \circ R\|_{\mathrm{op}} \overset{(iii)}{\leqslant} 2\mathbb{E}\|A \circ R\|_{\mathrm{op}}$$

where $\circ$ is the entry-wise matrix multiplication operation, *(i)* comes from Jensen's inequality, *(ii)* occurs since $A - A'$ and $(A - A') \circ R$ are identically distributed and *(iii)* is the triangle inequality. Next, we obtain an upper bound onto $\mathbb{E}\|A \circ R\|_{\mathrm{op}}$.

We define the family of independent random variables $(\xi_{ij} : 1 \leqslant i \leqslant j \leqslant n)$ where for all $1 \leqslant i \leqslant j \leqslant n$

$$\xi_{ij} = \begin{cases} \frac{1}{\sqrt{|\mathbb{E}A_{ij}|}} & \text{with probability } \frac{\mathbb{E}A_{ij}}{2} \\ -\frac{1}{\sqrt{|\mathbb{E}A_{ij}|}} & \text{with probability } \frac{\mathbb{E}A_{ij}}{2} \\ 0 & \text{with probability } 1 - \mathbb{E}A_{ij}. \end{cases} \tag{54}$$

We also put $b_{ij} := \sqrt{|\mathbb{E}A_{ij}|}$ for all $1 \leqslant i \leqslant j \leqslant n$. It is straightforward to see that $(\xi_{ij}b_{ij} : 1 \leqslant i \leqslant j \leqslant n)$ and $(A_{ij}R_{ij} : 1 \leqslant i \leqslant j \leqslant n)$ have the same distribution. As a consequence, $\|A \circ R\|_{\mathrm{op}}$ and $\|X\|_{\mathrm{op}}$ have the same distribution where $X \in \mathbb{R}^{n\times n}$ is a symmetric matrix with $X_{ij} = \xi_{ij}b_{ij}$ for $1 \leqslant i \leqslant j \leqslant n$. An upper bound on $\mathbb{E}\|X\|_{\mathrm{op}}$ follow from the next result due to Bandeira and Van Handel (2016).

Theorem 9 (Corollary 3.6 in Bandeira and Van Handel (2016)) *Let* $\xi_{ij}, 1 \leqslant i \leqslant j \leqslant n$ *be independent symmetric random variables with unit variance and* $(b_{ij}, 1 \leqslant i \leqslant j \leqslant n)$ *be a family of real numbers. Let* $X \in \mathbb{R}^{n\times n}$ *be the random symmetric matrix defined by* $X_{ij} = \xi_{ij}b_{ij}$ *for all* $1 \leqslant i \leqslant j \leqslant n$. *Let* $\sigma := \max_{1\leqslant i\leqslant n}\left\{\sqrt{\sum_{j=1}^{n} b_{ij}^2}\right\}$. *Then, for any* $\alpha \geqslant 3$,

$$\mathbb{E}\|X\|_{op} \leqslant e^{\frac{2}{\alpha}}\left[2\sigma + 14\alpha \max_{1\leqslant i\leqslant j\leqslant n}\left\{\|\xi_{ij}b_{ij}\|_{2\lceil\alpha\log(n)\rceil}\right\}\sqrt{\log(n)}\right]$$

where, for any $q > 0$, $\|\cdot\|_q$ *denotes the* L_q-norm.

Since $(\xi_{ij} : 1 \leqslant i \leqslant j \leqslant n)$ are independent symmetric such that $\mathrm{Var}(\xi_{ij}) = \mathbb{E}[\xi_{ij}^2] = 1$ we can apply Lemma 9 to upper bound $\mathbb{E}\|X\|_{\mathrm{op}} = \mathbb{E}\|A \circ R\|_{\mathrm{op}}$. We have $\|\xi_{ij}b_{ij}\|_{2\lceil\alpha\log(n)\rceil} \leqslant 1$ for any $\alpha \geqslant 3$ and $b_{ij}^2 = |\mathbb{E}A_{ij}| \leqslant \delta\nu$. It therefore follows from Lemma 9 for $\alpha = 3$ that

$$\mathbb{E}\|W\|_{\mathrm{op}} \leqslant 2e^{\frac{2}{3}}\left[2\sqrt{n\delta\nu} + 42\sqrt{\log(n)}\right] \leqslant 8\sqrt{n\delta\nu} + 168\sqrt{\log(n)}. \tag{55}$$

The final step to prove Lemma 3 is a concentration argument showing that $\|W\|_{\mathrm{op}}$ is close to its expectation with high-probability. To that end we rely on a general result for Lipschitz and separately convex functions from Boucheron et al. (2013). We first recall that a real-valued function f of N variables is said separately convex when for every $i = 1, \ldots, N$ it is a convex function of the i-th variable if the rest of the variables are fixed.

Theorem 10 (Theorem 6.10 in Boucheron et al. (2013)) *Let $\mathcal{X}$ be a convex compact set in $\mathbb{R}$ with diameter B. Let $X_1, \cdots, X_N$ be independent random variables taking values in $\mathcal{X}$. Let $f : \mathcal{X}^N \to \mathbb{R}$ be a separately convex and 1-Lipschitz function, w.r.t. the ℓ_2^N-norm (i.e. $|f(x) - f(y)| \leqslant \|x - y\|_2$ for all $x, y \in \mathcal{X}^N$). Then $Z = f(X_1, \ldots, X_N)$ satisfies, for all $t > 0$, with probability at least $1 - \exp(-t^2/B^2)$, $Z \leqslant \mathbb{E}[Z] + t$.*

We apply Theorem 10 to $Z := \|W\|_{\mathrm{op}} = f(A_{ij} - \mathbb{E}A_{ij}, 1 \leqslant i \leqslant j \leqslant n) = \frac{1}{\sqrt{2}} \|A - \mathbb{E}A\|_{\mathrm{op}}$ where f is a 1-Lipschitz w.r.t. ℓ_2^N-norm for $N = n(n-1)/2$ and separately convex function and $(A_{ij} - \mathbb{E}A_{ij}, 1 \leqslant i \leqslant j \leqslant n)$ is a family of N independent random variables. Moreover, for each $i \geqslant j$, $(A - \mathbb{E}A)_{ij} \in [-1 - \delta(2p-1), 1 - \delta(2q-1)]$, which is a convex compact set with diameter $B = 2(1 + \delta(p - q)) \leqslant 4$. Therefore, it follows from Theorem 10 that for all $t > 0$, with probability at least $1 - \exp(-t^2/16)$, $\|W\|_{\mathrm{op}} \leqslant \mathbb{E}\|W\|_{\mathrm{op}} + \sqrt{2}t$. In particular, we finish the proof of Lemma 3 for $t = 4\sqrt{\delta\nu n}$ and using the bound from (55). ∎

It follows from Lemma 3 that when $n\nu\delta \geqslant \log n$, $\|W\|_{\mathrm{op}} \leqslant 184\sqrt{n\delta\nu}$ with probability at least $1 - \exp(-\delta\nu n)$. Using this later result together with Lemma 2 concludes the proof of Proposition 3.

Appendix C. Proof of Theorem 6 for Angular Synchronization with additive noise

Here we consider the following model: we observe $C = Z^* + \sigma W$, where $Z^* = x^*(\overline{x^*})^\top$, $x^* = (e^{\iota\theta_i})_{i=1}^n$ and $W \in \mathbb{C}^{n \times n}$ is a complex Wigner matrix (i.e. $W = \overline{W}^T$, its above-diagonal entries are complex numbers whose real and imaginary parts are independent normally distributed random variables with mean zero and variance $1/2$, and its diagonal entries are zero).

Let us first show that Z^* is the oracle in our approach. That is to show that

$$Z^* \in \underset{Z \in \mathcal{C}}{\mathrm{argmax}} \langle \mathbb{E}C, Z \rangle \tag{56}$$

where $\mathcal{C} := \{Z \in \mathbb{H}_n : Z \succeq 0, \mathrm{diag}(Z) = \mathbf{1}_n\}$.

We recall that the offsets are $\delta_{ij} = \theta_i - \theta_j[2\pi]$ for $i, j = 1, \ldots, n$. Let $\gamma_1, \ldots, \gamma_n \in [0, 2\pi[$ and define $x_i = e^{\iota\gamma_i}, i = 1, \ldots, n$. We have for all $i, j = 1, \ldots, n$,

$$\gamma_i - \gamma_j = \delta_{ij}[2\pi] \iff e^{\iota\gamma_i} - e^{\iota\delta_{ij}}e^{\iota\gamma_j} = 0 \iff Z^*_{ij}x_j - x_i \iff \mathbb{E}C_{ij}x_j - x_i = 0$$

It follows that

$$\underset{x \in \mathbb{C}^n : |x_i| = 1 \forall i}{\mathrm{argmin}} \left\{ \sum_{i,j=1}^n |\mathbb{E}C_{ij}x_j - x_i|^2 \right\} = \left\{ (e^{\iota(\theta_i + \theta_0)})_{i=1}^n : \theta_0 \in [0, 2\pi) \right\}. \tag{57}$$

Let us now rewrite the latter optimization problem as a SDP problem.

Let $x \in \mathbb{C}^n$ be such that $|x_i| = 1$ for all $i = 1, \ldots, n$. We have

$$\sum_{i,j=1}^{n} |\mathbb{E}C_{ij}x_j - x_i|^2 = \sum_{i,j=1}^{n} (\mathbb{E}C_{ij}x_j - x_i)\overline{(\mathbb{E}C_{ij}x_j - x_i)}$$

$$= \sum_{i,j=1}^{n} \left[\underbrace{|\mathbb{E}C_{ij}x_j|^2}_{=1} + \underbrace{|x_i|}_{=1}^2 - (\mathbb{E}C_{ij}x_j + \overline{\mathbb{E}C_{ij}x_j\bar{x}_i}) \right] = 2n^2 - 2\Re\left(\sum_{i,j=1}^{n} \mathbb{E}C_{ij}x_j\bar{x}_i \right)$$

$$= 2n^2 - 2\Re\left(\bar{x}^\top \mathbb{E}Cx \right) = 2n^2 - \bar{x}^\top \mathbb{E}Cx,$$

where we used in the last inequality that $\mathbb{E}[C] = Z^* = x^*\overline{x^*}^\top$ so that $\bar{x}^\top \mathbb{E}Cx = |\langle x, x^* \rangle|^2 \in \mathbb{R}$. Next, we see that $\bar{x}^\top \mathbb{E}Cx = \langle \mathbb{E}C, X \rangle$ where $X = x\bar{x}^\top$, hence, minimizing $x \in \mathbb{C}^n \to \sum_{i,j=1}^{n} |\mathbb{E}[C]_{ij}x_j - x_i|^2$ over all $x \in \mathbb{C}^n$ such that $|x_i| = 1$ boils down to maximizing $X \in \mathbb{C}^{n \times n} \to \mathrm{tr}(\mathbb{E}[C]X) \in \mathbb{R}$ over the set $\{X = x\bar{x}^\top : x \in \mathbb{C}^n, |x_i| = 1 \forall i\} = \{X \in \mathbb{H}_n : X \geq 0, \mathrm{diag}(X) = \mathbf{1}_n, \mathrm{rank}(X) = 1\}$ where $\mathbb{H}_n$ is the set of hermitian matrices of size $n \times n$. Then, it follows from (57) that

$$\underset{X \in \mathbb{H}_n : X \geq 0, \mathrm{diag}(X)=\mathbf{1}_n, \mathrm{rank}(X)=1}{\mathrm{argmax}} \langle \mathbb{E}C, X \rangle = \{Z^*\}$$

because for all $\theta_0 \in [0, 2\pi)$, $(e^{\iota(\theta_i+\theta_0)})_{i=1}^{n} \overline{(e^{\iota(\theta_i+\theta_0)})_{i=1}^{n}}^\top = Z^*$. The latter inequality is almost the result that we want to prove in (56); it only remains to show that the rank one constraint may be dropped. We use the same approach as the one we used to show (18).

First, one can easily check that the following inclusion holds true

$$\mathcal{C} \subset \mathcal{C}' := \{Z \in \mathbb{C}^{n \times n} : |Z_{ij}| \leq 1, \forall i, j \in [n]\}.$$

Second, the objective function $Z \to \Re(\langle \mathbb{E}C, Z \rangle)$ is linear (w.r.t. to real coefficient), hence, maximizing it over the convex set $\mathcal{C}'$ yields a solution at an extreme point of $\mathcal{C}'$, that is in the set of matrices $Z \in \mathbb{C}^{n \times n}$ such that $|Z_{ij}| = 1$ for all (i, j). Let $X = (e^{\iota\beta_{ij}})_{i,j \leq n}$ with $0 \leq \beta_{ij} < 2\pi$ be an extreme point of $\mathcal{C}'$. We have

$$\Re(\langle \mathbb{E}C, X \rangle) = \Re\left(\sum_{i,j=1}^{n} \mathbb{E}C_{ij}\overline{X_{ij}} \right) = \Re\left(\sum_{i,j=1}^{n} e^{\iota\delta_{ij}}e^{-\iota\beta_{ij}} \right) = \sum_{i,j=1}^{n} \cos(\delta_{ij} - \beta_{ij}) \leq n^2$$

and the maximum is achieved when $\beta_{ij} = \delta_{ij}[2\pi]$ for all (i, j), that is when $X = Z^*$. Since $Z^* \in \mathcal{C} \subset \mathcal{C}'$, then Z^* is the unique maximizer of $Z \to \langle \mathbb{E}C, Z \rangle$ over $\mathcal{C}$. Therefore (56) holds and so Z^* is also the oracle in this model.

Curvature of the objective function

Let $Z \in \mathcal{C}$. We can write $Z = (x_{ij}e^{\iota\beta_{ij}})_{i,j=1,\ldots,n}$, with $0 \leq x_{ij} \leq 1$ (we recall that $\mathcal{C} \subset \mathcal{C}'$ defined above) and $\beta_{ij} \in [0, 2\pi)$. On one side, we have

$$\langle \mathbb{E}C, Z^* - Z \rangle = \Re(\langle \mathbb{E}C, Z^* - Z \rangle) = \Re\left(\sum_{i,j=1}^{n} \mathbb{E}C_{ij}\overline{(Z^* - Z)_{ij}} \right)$$

$$= \Re\left(\sum_{i,j=1}^{n} e^{\iota\delta_{ij}}(e^{-\iota\delta_{ij}} - x_{ij}e^{-\iota\beta_{ij}}) \right) = \Re\left(\sum_{i,j=1}^{n} (1 - x_{ij}e^{\iota(\delta_{ij}-\beta_{ij})}) \right) = \sum_{i,j=1}^{n} (1 - x_{ij}\cos(\delta_{ij} - \beta_{ij})).$$

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On the other side, we have

$$
\begin{aligned}
\|Z^* - Z\|_2^2 &= \sum_{i,j=1}^n |(Z^* - Z)_{ij}|^2 = \sum_{i,j=1}^n |e^{\iota \delta_{ij}} - x_{ij} e^{\iota \beta_{ij}}|^2 = \sum_{i,j=1}^n |1 - x_{ij} e^{\iota(-\delta_{ij} + \beta_{ij})}|^2 \\
&= \sum_{i,j=1}^n (1 - x_{ij} \cos(\beta_{ij} - \delta_{ij}))^2 + x_{ij}^2 \sin^2(\beta_{ij} - \delta_{ij}) = \sum_{i,j=1}^n (1 - 2x_{ij} \cos(\beta_{ij} - \delta_{ij}) + x_{ij}^2) \\
&\leqslant 2 \sum_{i,j=1}^n (1 - x_{ij} \cos(\beta_{ij} - \delta_{ij})) = 2\langle \mathbb{E} C, Z^* - Z \rangle
\end{aligned}
$$

where we used in the last but one inequality that $0 \leqslant x_{ij} \leqslant 1$.

We conclude that the excess risk satisfies the following curvature: for all $Z \in \mathcal{C}$,

$$
\langle \mathbb{E} C, Z^* - Z \rangle \geqslant \frac{1}{2} \|X^* - X\|_2^2. \tag{58}
$$

That is Assumption 1 holds with $G : M \in \mathbb{C}^{n \times n} \to (1/2) \|M\|_2^2$.

Three upper bounds on the fixed point $r_G^*(\Delta)$ in the angular group synchronization model with additive noise

According to our methodology (associated with Corollary 1), we need to calculate the following fixed point: let $\Delta \in (0,1)$, and consider

$$
\begin{aligned}
r_G^*(\Delta) &= \inf \left\{ r > 0 : \mathbb{P} \left(\sup_{Z \in \mathcal{C}: G(Z^* - Z) \leqslant r} \langle C - \mathbb{E} C, Z - Z^* \rangle \leqslant r/2 \right) \geqslant 1 - \Delta \right\} \\
&= \inf \left\{ r > 0 : \mathbb{P} \left(\sup_{Z \in \mathcal{C}_r} \langle \sigma W, Z - Z^* \rangle \leqslant r/2 \right) \geqslant 1 - \Delta \right\}
\end{aligned}
$$

where $\mathcal{C}_r := \left\{ Z \in \mathcal{C} : \|Z - Z^*\|_2 \leqslant \sqrt{2r} \right\}$.

For pedagogical purposes, we show how to perform this computation via three different means, yielding three different results. We obtain the following upper bounds

$$
r_G^*(\exp(-n^2/2)) \leqslant 32\sigma^2 n^2, \, r_G^*(\exp(-n/2)) \leqslant 36 K_G^{\mathbb{C}} \sigma n^{3/2} \text{ and } r_G^*(5 \exp(-n/2)) \leqslant 2(20\sigma)^2 n.
$$

Each of the three bounds follows from a different strategy. The first one is based on the inclusion $\mathcal{C}_r \subset Z^* + \sqrt{2r} B_2^{n \times n}$, the second one on $\mathcal{C}_r \subset \mathcal{C}$ and is therefore the approach that we called "global", and the last one follows from the strategy used in Fei and Chen (2019) that we already used for the signed clustering problem.

FIRST UPPER BOUND ON THE FIXED POINT r_G^* USING $\mathcal{C}_r \subset Z^* + \sqrt{2r} B_2^{n \times n}$

In this section, we use the following inclusion to obtain the result

$$
\mathcal{C}_r \subset Z^* + \sqrt{2r} B_2^{n \times n}, \tag{59}
$$

where $B_2^{n \times n}$ is the Euclidean ball in $\mathbb{R}^{n \times n}$. We have

$$
\sup_{Z \in \mathcal{C}: \|Z - Z^*\|_2 \leqslant \sqrt{2r}} \langle \sigma W, Z - Z^* \rangle \leqslant \sigma \sup_{Z \in \sqrt{2r} B_2^{n \times n}} \langle W, Z \rangle = \sigma \sqrt{2r} \|W\|_2.
$$

Next, we use Borell's concentration inequality for Gaussian processes Ledoux (2001): for all $u > 0$, with probability at least $1 - \exp(-u^2/2)$, we have that

$$\|W\|_2 \leqslant \mathbb{E}\|W\|_2 + \sup_{Z \in B_2^{n \times n}} \sqrt{\mathbb{E}\langle W, Z\rangle^2} u \leqslant n + u.$$

As a consequence, for $\Delta = \exp(-n^2/2)$ and $r = 32\sigma^2 n^2$, we have, with probability at least $1 - \Delta$,

$$\sup_{Z \in \mathcal{C}: \|Z-Z^*\|_2 \leqslant \sqrt{2r}} \langle \sigma W, Z - Z^*\rangle \leqslant 2n\sigma\sqrt{2r} \leqslant r/2.$$

Hence, one has $r_G^*(\Delta) \leqslant 32\sigma^2 n^2$.

We apply Corollary 1 to get that with probability at least $1 - \exp(-n^2/2)$

$$\frac{1}{2}\left\|\tilde{Z} - Z^*\right\|_2^2 \leqslant \langle \mathbb{E}C, Z^* - \tilde{Z}\rangle \leqslant r_G^*(\exp(-n^2/2)) \leqslant 32\sigma^2 n^2. \tag{60}$$

Next, we know that the oracle Z^* is the rank-one matrix $x^*\overline{x^*}^\top$ which has n as its largest eigenvalue and associated eigenspace $\{\lambda z^* : \lambda \in \mathbb{C}\}$. In particular, Z^* has a spectral gap $g = n$. Let $\bar{x} \in \mathbb{C}^n$ be a top eigenvector of $\tilde{Z}$ with norm $\|\bar{x}\|_2 = \sqrt{n}$. It follows from Davis-Kahan Theorem (see, for example, Theorem 4.5.5 in Vershynin (2018) or Theorem 4 in Vu (2010)) that there exists an universal constant $c_0 > 0$ such that

$$\min_{z \in \mathbb{C}: |z|=1} \left\|\frac{\bar{x}}{\sqrt{n}} - z\frac{x^*}{\sqrt{n}}\right\|_2 \leqslant \frac{c_0}{g}\left\|\tilde{Z} - Z^*\right\|_2,$$

where $g = n$ is the spectral gap of Z^*. Using (61), we conclude that, with probability at least $1 - \exp(-n^2/2)$, it holds true that

$$\min_{z \in \mathbb{C}: |z|=1} \|\bar{x} - zx^*\|_2 \leqslant 8c_0\sigma\sqrt{n}.$$

SECOND UPPER BOUND ON THE FIXED POINT r_G^*: THE GLOBAL APPROACH

One may wonder what type of result we can get for the angular group synchronization problem with additive noise using the global approach. It is the aim of this last section to answer this question.

As for the community detection problem we will use Grothendieck inequality for the global approach. As in (59), the global approach is also using an inclusion of the localized set $\mathcal{C}_r$, but unlike (59), we just drop off the localization: we are simply using $\mathcal{C}_r \subset \mathcal{C}$. We have that

$$\sup_{Z \in \mathcal{C}: \|Z-Z^*\|_2 \leqslant \sqrt{2r}} \langle \sigma W, Z - Z^*\rangle \leqslant \sigma \sup_{Z \in \mathcal{C}} \langle W, Z - Z^*\rangle \leqslant 2K_G^{\mathbb{C}}\sigma\|W\|_{cut},$$

where we used Grothendieck's inequality as in (42) but in the complex case ($K_G^{\mathbb{C}}$ denoting Grothendieck's constant in the complex case). Here the cut norm in the complex case is defined as

$$\|W\|_{cut} = \sup_{s_i, t_j \in \mathbb{C}: |s_i|=|t_j|=1} \left|\sum_{i,j} W_{ij}s_i t_j\right|.$$

We therefore end up with the computation of the cut norm of the noise as in Guédon and Vershynin (2016) for the community detection problem.

Here the noise being Gaussian, the cut norm $\|W\|_{cut}$ is the supremum of a Gaussian process for which we can use Borell's concentration inequality (see Ledoux (2001)) to get for all $u > 0$, with probability at least $1 - \exp(-u^2/2)$,

$$\|W\|_{cut} \leqslant \mathbb{E}\,\|W\|_{cut} + u \sup_{s_i, t_j \in \mathbb{C} : |s_i| = |t_j| = 1} \sqrt{\mathbb{E}\Big|\sum_{i,j} W_{ij} s_i t_j\Big|^2} \leqslant \|W\|_{cut} + un.$$

Next, we use Slepian's lemma (see Chapter 3 from Ledoux and Talagrand (1991)) to handle the complexity term $\mathbb{E}\,\|W\|_{cut}$. First, we need to upper bound the canonical metric associated with the Gaussian process: for every $s_i, s_i', t_j, t_j' \in \mathbb{C} : |s_i| = |t_j| = |s_i'| = |t_j'| = 1$ we have

$$\mathbb{E}\Big|\sum_{i,j} W_{ij} s_i t_j - \sum_{i,j} W_{ij} s_i' t_j'\Big|^2 = \sum_{i,j} |s_i t_j - s_i' t_j'|^2$$

$$= \sum_{i,j} |s_i(t_j - t_j') - (s_i' - s_i)t_j'|^2 \leqslant 2n\left(\sum_j |t_j - t_j'|^2 + \sum_i |s_i - s_i'|^2\right)$$

$$= 2n\mathbb{E}\Big|\sum_i g_i(s_i - s_i') + \sum_j \eta_j(t_j - t_j')\Big|^2,$$

where $(g_i)_i, (\eta_j)_j$ are i.i.d. $\mathcal{N}(0, 1)$. It follows from Slepian's lemma that

$$\mathbb{E}\,\|W\|_{cut} \leqslant \sqrt{2n}\,\mathbb{E} \sup_{s_i, t_j \in \mathbb{C} : |s_i| = |t_j| = 1} \Big|\sum_i g_i(s_i - s_i') + \sum_j \eta_j(t_j - t_j')\Big| \leqslant 4\sqrt{2n}\,n.$$

Together with Borell's inequality above for $u = \sqrt{n}$, we obtain that with probability at least $1 - \exp(-n/2)$, $\|W\|_{cut} \leqslant 9n^{3/2}$.

As a consequence, for $\Delta' = \exp(-n/2)$, we have $r_G^*(\Delta') \leqslant 36K_G^{\mathbb{C}}\sigma n^{3/2}$. It follows from Corollary 1 that with probability at least $1 - \exp(-n/2)$

$$\frac{1}{2}\left\|\tilde{Z} - Z^*\right\|_2^2 \leqslant \langle \mathbb{E}C, Z^* - \tilde{Z}\rangle \leqslant r_G^*(\exp(-n/2)) \leqslant 36K_G^{\mathbb{C}}\sigma n^{3/2}. \tag{61}$$

and so

$$\min_{z \in \mathbb{C} : |z| = 1} \|\bar{x} - zx^*\|_2 \leqslant c_0\sqrt{36K_G^{\mathbb{C}}\sigma}\,n^{1/4}.$$

We conclude that the global approach is better than the local approach using the inclusion in (59).

THIRD UPPER BOUND ON THE FIXED POINT r_G^*: END OF THE PROOF OF THEOREM 6

The final approach is based on a decomposition from Fei and Chen (2019) that we already used for the signed clustering problem. Here, for the angular group synchronization problem, the projection operator is simpler since Z^* is the rank-one matrix $x^*\overline{x^*}^\top$ and all the processes are Gaussian processes.

In order to work with independent random variables, we consider the following matrix $\Psi \in \mathbb{C}^{n \times n}$

$$\Psi_{ij} = \begin{cases} W_{ij} \text{ if } i \leqslant j \\ 0 \text{ otherwise,} \end{cases} \tag{62}$$

where 0 entries are considered as independent Gaussian variables with 0 variance and therefore, Ψ has independent Gaussian entries, and satisfies the relation $W = \Psi + \overline{\Psi}^\top$.

Like we did for the signed clustering problem, we decompose the inner product $\langle W, Z - Z^* \rangle$ into two parts according to the SVD of Z^*. We know that Z^* is the rank-one matrix $Z^* = x\overline{x^*}^\top$, then $v := x^*/\sqrt{n}$ is a unit singular vector of Z^* and we define the following projection operator

$$\mathcal{P} : M \in \mathbb{C}^{n \times n} \to v\bar{v}^\top M + Mv\bar{v}^\top - v\bar{v}^\top Mv\bar{v}^\top,$$

and its associated orthogonal projection

$$\mathcal{P}^\perp : M \in \mathbb{C}^{n \times n} \to M - \mathcal{P}(M) = (I_n - v\bar{v}^\top)M(I_n - v\bar{v}^\top).$$

For any $Z \in \mathcal{C}_r := \mathcal{C} \cap \{Z^* + \sqrt{2r}B_2^{n \times n}\}$, we consider the following decomposition as in Fei and Chen (2019)

$$\langle W, Z - Z^* \rangle = \underbrace{\langle \mathcal{P}(Z - Z^*), W \rangle}_{S_1(Z)} + \underbrace{\langle \mathcal{P}^\perp(Z - Z^*), W \rangle}_{S_2(Z)}.$$

Next, we upper bound with large probability each of the two last terms uniformly over all $Z \in \mathcal{C}_r$. We start with the $S_1(Z)$ term: for any $Z \in \mathcal{C}_r$, we have

$$\begin{aligned} S_1(Z) &= \langle W, \mathcal{P}(Z - Z^*) \rangle = \langle \mathcal{P}(W), Z - Z^* \rangle \\ &= \langle v\bar{v}^\top W, Z - Z^* \rangle + \langle Wv\bar{v}^\top, Z - Z^* \rangle - \langle v\bar{v}^\top Wv\bar{v}^\top, Z - Z^* \rangle \\ &= 2\langle v\bar{v}^\top W, Z - Z^* \rangle - \langle v\bar{v}^\top Wv\bar{v}^\top, Z - Z^* \rangle \\ &= 2\langle v\bar{v}^\top \Psi, Z - Z^* \rangle + 2\langle v\bar{v}^\top \overline{\Psi}^\top, Z - Z^* \rangle - \langle v\bar{v}^\top (\Psi + \overline{\Psi}^\top)v\bar{v}^\top, Z - Z^* \rangle \\ &= 2\langle v\bar{v}^\top \Psi, Z - Z^* \rangle + 2\langle v\bar{v}^\top \bar{\Psi}^\top, Z - Z^* \rangle - 2\langle v\bar{v}^\top \Psi v\bar{v}^\top, Z - Z^* \rangle \\ &= 2\langle v\bar{v}^\top \Psi, Z - Z^* \rangle + 2\langle v\bar{v}^\top \bar{\Psi}^\top, Z - Z^* \rangle - 2\langle v\bar{v}^\top \Psi, (Z - Z^*)v\bar{v}^\top \rangle. \end{aligned}$$

Then, bounding separately each of those three terms will lead us to a bound for $S_1(Z)$. Let us show how to bound the first term. Similar arguments can be used to control the other two terms.

We define $V := v\bar{v}^\top \Psi$, so that $V_{ij} = \sum_k v_i \bar{v}_k \Psi_{kj} = \sum_{k \leqslant j} v_i \bar{v}_k W_{kj}$. We want to find a high-probability upper bound on $\langle V, Z - Z^* \rangle$. To that end we simply use the inclusion $\mathcal{C}_r \subset Z^* + \sqrt{2r}B_2^{n \times n}$ to get

$$\sup_{Z \in \mathcal{C}_r} \langle V, Z - Z^* \rangle \leqslant \sup_{Z \in \sqrt{2r}B_2^{n \times n}} \langle V, Z \rangle = \sqrt{2r}\, \|V\|_2$$

so that

$$\mathbb{E}\left[\sup_{Z\in\mathcal{C}_r}\langle V,Z-Z^*\rangle\right] \leqslant \sqrt{2r}\mathbb{E}\|V\|_2 \leqslant \sqrt{2r}\left(\sum_{i,j}\sum_{k\leqslant j}|v_i|^2|v_j|^2\mathbb{E}|\Psi_{kj}|^2\right)^{\frac{1}{2}}$$

$$= \sqrt{2r}\left(\frac{1}{n}\sum_j j\right)^{1/2} \leqslant \sqrt{2rn}.$$

Moreover, we have

$$\mathbb{E}[|\langle V,Z-Z^*\rangle|^2] = \mathbb{E}[|\langle \Psi, v\bar{v}^T(Z-Z^*)\rangle|^2] = \sum_{i,j}|(v\bar{v}^T(Z-Z^*))_{ij}|^2$$

$$= \sum_{i,j}\left|\sum_k v_i\bar{v}_k(Z-Z^*)_{kj}\right|^2 \leqslant \frac{1}{n^2}\sum_{i,j}\sum_k|(Z-Z^*)_{kj}|^2 \leqslant \frac{\|Z-Z^*\|_2^2}{n} \leqslant \frac{2r}{n}.$$

Now, we apply Borell's inequality (see, for instance, Theorem 4.1 in Ledoux (2001) or page 56-57 in Ledoux and Talagrand (1991)): for $u = n$, with probability at least $1 - \exp(-u^2/2)$,

$$\sup_{Z\in\mathcal{C}_r}\langle V,Z-Z^*\rangle \leqslant \mathbb{E}\left[\sup_{Z\in\mathcal{C}_r}\langle V,Z-Z^*\rangle\right] + \sup_{Z\in\mathcal{C}_r}\sqrt{\mathbb{E}\left[|\langle V,Z-Z^*\rangle|^2\right]}u$$

$$\leqslant \sqrt{2rn} + \sqrt{\frac{2r}{n}}u = 2\sqrt{2rn}.$$

Similar calculus yield to the same upper bounds for the two other terms. Therefore, we obtain, with probability at least $1 - 3\exp(-n^2/2)$, it holds true that

$$\sup_{Z\in\mathcal{C}_r} S_1(Z) \leqslant 6\sqrt{2rn}. \tag{63}$$

Now, it remains to control the second $S_2(Z)$ term. For any $Z \in \mathcal{C}_r$, we have

$$S_2(Z) = \langle W, \mathcal{P}^\perp(Z-Z^*)\rangle = \langle W, \mathcal{P}^\perp(Z)\rangle \leqslant \|\mathcal{P}^\perp(Z)\|_* \|W\|_{\mathrm{op}}.$$

Since $Z \succeq 0$, we have $(I_n - v\bar{v}^\top)Z(I_n - v\bar{v}^\top) \succeq 0$, that is $\mathcal{P}^\perp(Z) \succeq 0$ and so

$$\|\mathcal{P}^\perp(Z)\|_* = \mathrm{Tr}(\mathcal{P}^\perp(Z)) \overset{(i)}{=} \mathrm{Tr}(\mathcal{P}^\perp(Z-Z^*)) = \mathrm{Tr}((I_n - v\bar{v}^\top)(Z-Z^*)(I_n - v\bar{v}^\top))$$

$$\overset{(ii)}{=} \mathrm{Tr}((I_n - v\bar{v}^\top)(Z-Z^*)) = \mathrm{Tr}(Z-Z^*) - \mathrm{Tr}(v\bar{v}^\top(Z-Z^*))$$

$$\overset{(iii)}{=} \mathrm{Tr}(v\bar{v}^\top(Z^*-Z)) = \sum_{i,j}v_i\bar{v}_j(Z^*-Z)_{ij} \leqslant \|v\|_2^2\|Z-Z^*\|_2$$

$$= \|Z-Z^*\|_2 \leqslant \sqrt{2r}.$$

where (i) is due to the fact that $\mathcal{P}^\perp(Z^*) = 0$ by construction, (ii) holds because $(I_n - v\bar{v}^\top)$ is Hermitian and $(I_n - v\bar{v}^\top)^2 = (I_n - v\bar{v}^\top)$ and (iii) holds since $\mathrm{Tr}(Z) = \mathrm{Tr}(Z^*) = 1$ for any $Z \in \mathcal{C}_r$.

Moreover, it follows from Davidson and Szarek (2001) that, for all $u > 0$, with probability at least $1 - 2\exp(-u^2/2)$, $\|W\|_{\mathrm{op}} \leqslant 2\sqrt{n} + u$. We conclude that with probability at least $1 - 2\exp(-2n)$, we have

$$\sup_{Z \in \mathcal{C}_r} S_2(Z) \leqslant 4\sqrt{2nr}. \tag{64}$$

It follows from (63) and (64) that with probability at least $1 - 5\exp(-n/2)$

$$\sup_{Z \in \mathcal{C}_r} \langle W, Z - Z^* \rangle \leqslant 10\sqrt{2rn}.$$

Then, for $\Delta = 5\exp(-n/2)$ and $r = 2(20\sigma)^2 n$ we have, with probability at least $1 - \Delta$, $\sup_{Z \in \mathcal{C}_r} \langle \sigma W, Z - Z^* \rangle \leqslant r/2$. Hence, we conclude that $r_G^*(\Delta) \leqslant 2(20\sigma)^2 n$.

Now, we apply Corollary 1 to get that, with probability at least $1 - 5\exp(-n/2)$

$$\frac{1}{2} \left\| \tilde{Z} - Z^* \right\|_2^2 \leqslant 2(20\sigma)^2 n.$$

Next, we know that the oracle Z^* is the rank-one matrix $x^* \overline{x^*}^\top$ which has n for largest eigenvalue and associated eigenspace $\{\lambda z^* : \lambda \in \mathbb{C}\}$. In particular, Z^* has a spectral gap $g = n$. Let $\bar{x} \in \mathbb{C}^n$ be a top eigenvector of $\tilde{Z}$ with norm $\|\bar{x}\|_2 = \sqrt{n}$.

It follows from Davis-Kahan Theorem (see, for example, Theorem 4.5.5 in Vershynin (2018) or Theorem 4 in Vu (2010)) that there exists an universal constant $c_0 > 0$ such that

$$\min_{z \in \mathbb{C}: |z|=1} \left\| \frac{\bar{x}}{\sqrt{n}} - z \frac{x^*}{\sqrt{n}} \right\|_2 \leqslant \frac{c_0}{g} \left\| \hat{Z} - Z^* \right\|_2,$$

where $g = n$ is the spectral gap of Z^*. Using the previous inequality, we conclude that, with probability at least $1 - 5\exp(-n/2)$

$$\min_{z \in \mathbb{C}: |z|=1} \|\bar{x} - z x^*\|_2 \leqslant 40 c_0 \sigma.$$

Appendix D. Solving SDPs in practice

The practical implementation of our approach to the problems of synchronization, signed clustering and MAX-CUT resort to solving a convex optimization problem. In the present section, we describe the various algorithms we used for solving these SDP's.

Pierra's method

For SDP's with constraints on the entries, we propose a simple modification of the method initially proposed by Pierra in Pierra (1984). Let $f \colon \mathbb{R}^{n \times n} \mapsto \mathbb{R}$ be a convex function. Let $\mathcal{C}$ denote a convex set which can be written as the intersection of convex sets $\mathcal{C} = \mathcal{S}_1 \cap \cdots \cap \mathcal{S}_J$. Let us define $\mathbb{H} = \mathbb{R}^{n \times n} \times \cdots \times \mathbb{R}^{n \times n}$ (J times) and let $\mathbb{D}$ denote the (diagonal) subspace of $\mathbb{H}$ of vectors of the form $(Z, \ldots, Z)$. In this new formalism, the problem can now be formulated as a minimization problem over the intersection of two sets only, i.e.

$$\min_{\mathsf{Z} \in \mathbb{H}} \left(\frac{1}{J} \sum_{j=1}^{J} f(\mathsf{Z}_j) : \mathsf{Z} = (\mathsf{Z}_j)_{j=1}^{J} \in (\mathcal{S}_1 \times \cdots \times \mathcal{S}_J) \cap \mathbb{D} \right).$$

Define $F(\mathsf{Z}) = \frac{1}{\mathsf{J}} \sum_{\mathsf{j}=1}^{\mathsf{J}} \mathsf{f}(\mathsf{Z}_\mathsf{j})$. The algorithm proposed by Pierra in Pierra (1984) consists of performing the following iterations

$$\mathsf{Z}^{p+1} = \mathrm{Prox}_{I_{\mathcal{S}_1 \times \cdots \times \mathcal{S}_J} + \frac{1}{2}\varepsilon F}\left(\mathsf{B}^p\right) \ \text{and} \ \mathsf{B}^{p+1} = \mathrm{Proj}_{\mathbb{D}}(\mathsf{Z}^{p+1}).$$

APPLICATION TO COMMUNITY DETECTION

Let us now present the computational details of Pierra's method for the community detection problem. We will estimate its membership matrix $\bar{Z}$ via the following SDP estimator

$$\hat{Z} \in \mathrm{argmax}_{Z \in \mathcal{C}} \langle A, Z \rangle,$$

where $\mathcal{C} = \{Z \in \mathbb{R}^{n \times n}, Z \succeq 0, Z \geqslant 0, \mathrm{diag}(Z) \preceq I_n, \sum Z_{ij} \leqslant \lambda\}$ and $\lambda = \sum_{i,j=1}^n \bar{Z}_{ij} = \sum_{k=1}^K |\mathcal{C}_k|^2$ denotes the number of nonzero elements in the membership matrix $\bar{Z}$. The motivation for this approach stems from the fact that the membership matrix $\bar{Z}$ is actually the oracle, i.e., $Z^* = \bar{Z}$, where $Z^* \in \mathrm{argmax}_{Z \in \mathcal{C}} \langle \mathbb{E}[A], Z \rangle$. The function f to minimize in the Pierra algorithm is defined as $f(Z) = -\langle A, Z \rangle$.

Let us denote by $\mathbb{S}_+$ the set of symmetric positive semi-definite matrices in $\mathbb{R}^{n \times n}$. The set $\mathcal{C}$ is the intersection of the sets

$$S_1 = \mathbb{S}_+; \qquad S_2 = \left\{Z \in \mathbb{R}^{n \times n} \mid Z \geqslant 0\right\}; \qquad S_3 = \left\{Z \in \mathbb{R}^{n \times n} \mid \mathrm{diag}(Z) \preceq I\right\};$$

$$\text{and } S_4 = \left\{Z \in \mathbb{R}^{n \times n} \mid \sum_{i,j=1}^n Z_{ij} \leqslant \lambda\right\}.$$

We now compute for all $\mathsf{B} = (\mathsf{B}_\mathsf{j})_{\mathsf{j}=1}^4 \in (\mathbb{R}^{\mathsf{n} \times \mathsf{n}})^4$ and $j = 1, \ldots, 4$ ($J = 4$ here)

$$\mathrm{Prox}_{I_{S_1 \times \cdots \times S_4} + \frac{1}{2}\varepsilon F}\left(\mathsf{B}\right)_j = \mathrm{Prox}_{I_{S_j} + \frac{1}{2J}\varepsilon f}\left(\mathsf{B}_j\right).$$

We have for $J = 4$

$$\mathrm{Prox}_{I_{S_j} + \frac{1}{2J}\varepsilon f}\left(\mathsf{B}_j\right) = \mathrm{argmin}_{Z \in S_j} \ -\frac{\epsilon}{2J}\langle A, Z \rangle + \frac{1}{2}\|Z - \mathsf{B}_j\|_F^2 = P_{S_j}\left(\mathsf{B}_j + \frac{\epsilon}{2J}A\right)$$

On the other hand, the projections operators $P_{S_j}, j = 1, 2, 3, 4$ are given by

$$P_{S_1}(\mathsf{Z}_1) = U \max\{\Sigma, 0\} U^\top, \quad \text{where } \mathsf{Z}_1 \text{ has SVD } \mathsf{Z}_1 = \mathsf{U}\Sigma\mathsf{U}^\top,$$
$$P_{S_2}(\mathsf{Z}_2) = \max\{\mathsf{Z}_2, 0\}, \qquad P_{S_3}(\mathsf{Z}_3) = \mathsf{Z}_3 - \mathrm{diag}(\mathsf{Z}_3) + \min\{1, \mathrm{diag}(\mathsf{Z}_3)\},$$
$$P_{S_4}(\mathsf{Z}_4) = \frac{\lambda}{\sum_{ij}(\mathsf{Z}_4)_{ij}} \mathsf{Z}_4.$$

To sum up, Pierra's method can be formulated as follows.

For all iterations k in $\mathbb{N}$, compute the SVD of $\mathsf{B}_1^k + \frac{\epsilon}{2\cdot 4}A = U^k \Sigma^k (U^k)^\top$. Then compute for all $j = 1, \ldots, 4$

$$\mathsf{B}_\mathsf{j}^{\mathsf{k}+1} = \frac{1}{4}\left(U^k \max\left\{\Sigma^k, 0\right\}(U^k)^\top + \max\left\{\mathsf{B}_2^k + \frac{\epsilon}{2\cdot 4}A, 0\right\} + \mathsf{B}_3^k + \frac{\epsilon}{2\cdot 4}A - \mathrm{diag}(\mathsf{B}_3^k + \frac{\epsilon}{2\cdot 4}A)\right.$$

$$\left. + \min\left\{1, \mathrm{diag}(\mathsf{B}_3^k + \frac{\epsilon}{2\cdot 4}A)\right\} + \frac{\lambda}{\sum_{ij}(\mathsf{B}_4^k + \frac{\epsilon}{2\cdot 4}A)_{ij}}\mathsf{B}_4^k + \frac{\epsilon}{2\cdot 4}A\right).$$

APPLICATION TO SIGNED CLUSTERING

Let us now turn to the signed clustering problem. We will estimate its membership matrix $\bar{Z}$ via the following SDP estimator $\hat{Z} \in \text{argmax}_{Z \in \mathcal{C}} \langle A, Z \rangle$, where $\mathcal{C} = \{Z \in \mathbb{R}^{n \times n} : Z \succeq 0, Z_{ij} \in [0, 1], Z_{ii} = 1, i = 1, \ldots, n\}$. As in the community detection case the function f to minimize in the Pierra algorithm is defined as $f(Z) = -\langle A, Z \rangle$.

Let us denote by $\mathbb{S}_+$ the set of symmetric positive semi-definite matrices in $\mathbb{R}^{n \times n}$. The set $\mathcal{C}$ is the intersection of the sets $S_1 = \mathbb{S}_+$, $S_2 = \{Z \in \mathbb{R}^{n \times n} \mid Z \in [0, 1]^{n \times n}\}$ and $S_3 = \{Z \in \mathbb{R}^{n \times n} \mid Z_{ii} = 1, \ i = 1, \ldots, n\}$.

As before, for $j = 1, \ldots, 3$

$$\text{Prox}_{I_{S_j} + \frac{1}{2 \cdot 3} \varepsilon f} (\mathsf{B_j}) = P_{S_j} \left(\mathsf{B_j} + \frac{\epsilon}{2 \cdot 3} \, A \right)$$

and the projection operators P_{S_j}, $j = 1, 2$ are given by

$$P_{S_1}(\mathsf{Z_1}) = \mathsf{U} \max \{\Sigma, \mathsf{0}\} \mathsf{U}^\top, \ \mathsf{P_{S_2}}(\mathsf{Z_2}) = \min \{\max \{\mathsf{Z_2}, \mathsf{0}\}, 1\} \ \text{and} \ \mathsf{P_{S_3}}(\mathsf{Z_3}) = \mathsf{Z_3} - \text{diag}(\mathsf{Z_3}) + \mathsf{I}$$

To sum up, Pierra's method can be formulated as follows.

At each iteration k, compute the SVD of $\mathsf{B_1^k} + \frac{\epsilon}{2 \cdot 3} A = U^k \Sigma^k (U^k)^\top$. Then compute for all $j = 1, \ldots, 3$

$$\mathsf{B}_j^{k+1} = \frac{1}{3} \left(U^k \max \left\{ \Sigma^k, 0 \right\} U^{k^t} + \min \left\{ \max \left\{ \mathsf{B_2^k} + \frac{\epsilon}{2 \cdot 3} \, A, 0 \right\}, 1 \right\} \right.$$

$$\left. + \mathsf{B_3^k} + \frac{\epsilon}{2 \cdot 3} \, A - \text{diag} \left(\mathsf{B_3^k} + \frac{\epsilon}{2 \cdot 3} \, A \right) + I \right).$$

The Burer-Monteiro approach and the Manopt Solver

To solve the MAX-CUT and Angular Synchronization problems we rely on MANOPT, a freely available MATLAB toolbox for optimization on manifolds Boumal et al. (2014). MANOPT runs the Riemannian Trust-Region method on corresponding Burer-Monteiro non-convex problem with rank bounded by p as follows. The Burer-Monteiro approach consists of replacing the optimization of a linear function $\langle A, Z \rangle$ over the convex set $\mathcal{Z} = \{Z \succeq 0 : \mathcal{A}(Z) = b\}$ with the optimization of the quadratic function $\langle AY, Y \rangle$ over the non-convex set $\mathcal{Y} = \{Y \in \mathbb{R}^{n \times p} : \mathcal{A}(YY^T) = b\}$.

In the context of the MAX-CUT problem, the Burer-Monteiro approach amounts to the following steps. Denoting by Z the positive semidefinite matrix $Z = zz^T$, note that both the cost function and the constraints lend themselves to be expressed linearly in terms of Z. Dropping the NP-hard rank-1 constraint on Z, we arrive at the well-known convex relaxation of MAX-CUT from Goemans and Williamson (1995)

$$\hat{Z} \in \underset{Z \in \mathcal{C}}{\text{argmin}} \langle A, Z \rangle \tag{65}$$

where $\mathcal{C} := \{Z \in \mathbb{R}^{n \times n} : Z \succeq 0, Z_{ii} = 1, \forall i = 1, \ldots, n\}$.

If a solution $\hat{Z}$ of this SDP has rank 1, then $\hat{Z} = z^* z^{*T}$ for some z^*, which then gives the optimal cut. Recall that in the general case of higher rank $\hat{Z}$,

$$\hat{Y} \in \underset{X \in \mathcal{B}}{\operatorname{argmin}} \langle AY, Y \rangle, \tag{66}$$

where $\mathcal{B} := \{Y \in \mathbb{R}^{n \times p} : \operatorname{diag}(YY^T) = 1\}$. Note that the constraint $\operatorname{diag}(YY^T) = 1$ requires each row of Y to have unit ℓ_2^p norm, rendering Y to be a point on the Cartesian product of n unit spheres $\mathcal{S}_2^{p-1}$ in $\mathbb{R}^p$, which is a smooth manifold. Also note that the search space of the SDP is compact, since all Z feasible for the SDP have identical trace equal to n.

If the convex set $\mathcal{Z}$ is compact, and m denotes the number of constraints, it holds true that whenever p satisfies $\frac{p(p+1)}{2} \geqslant m$, the two problems share the same global optimum Barvinok (1995); Burer and Monteiro (2005). Building on pioneering work of Burer and Monteiro (2005), Boumal et al. (2016) showed that if the set $\mathcal{Z}$ is compact and the set $\mathcal{Y}$ is a smooth manifold, then $\frac{p(p+1)}{2} \geqslant m$ implies that, for almost all cost matrices A, global optimality is achieved by any Y satisfying a second-order necessary optimality conditions. Following Boumal et al. (2016), *for $p = \lceil \sqrt{2n} \rceil$, for almost all matrices A, even though* (66) *is non-convex, any local optimum Y is a global optimum (and so is $Z = YY^T$), and all saddle points have an escape (the Hessian has a negative eigenvalues).* Note that for $p > n/2$ the same statement holds true for *all A*, and was previously established by Boumal (2015).